SUBMANIFOLDS OF MAXIMAL NULLITY IN SYMMETRIC SPACES

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Abstract. A non-totally-geodesic submanifold of relative nullity $n-1$ in a symmetric space M is a cylinder over one of the following submanifolds: a surface F^2 of nullity 1 in a totally geodesic submanifold $N^3 \subset M$ locally isometric to $S^2(c) \times \mathbb{R}$ or $H^2(c) \times \mathbb{R}$; a submanifold F^{k+1} spanned by a totally geodesic submanifold $F^k(c)$ of constant curvature moving by a special curve in the isometry group of M; a submanifold F^{k+1} of nullity k in a flat totally geodesic submanifold of M; a curve.

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1. Introduction

Let F^n be a submanifold in a Riemannian manifold M^{n+p}. For a point $x \in F$ denote L_x the maximal subspace in the tangent space T_xF annulating the second fundamental form of F.

Definition 1. [1] *Index of relative nullity* $\nu(x)$ at a point $x \in F$ is the dimension of the subspace L_x. The *index of relative nullity* of the submanifold $F \subset M$ is

$$\nu(F) = \inf_{x \in F} \nu(x).$$

Submanifolds of positive nullity were extensively studied by many authors (see the recent review [2]). The case $\nu = n$ is trivial: such a submanifold is totally geodesic. In this paper we study the local behaviour of submanifolds with $\nu = n - 1$; they will be called the submanifolds of the maximal nullity. The structure of such submanifolds is known if M is of a constant curvature ([1,3]). If F is smooth enough and is free of *planar points* (with $\nu(x) = n$), the following holds (see the Theorem A below):

1. F is of the same constant curvature;
2. The distribution $L = \cup_x L_x$ is integrable;
3. The integral foliation $\mathcal{L}$ of L is totally geodesic both in F and in M;
4. The submanifold F is either a union of subspaces of M tangent to osculating spaces of a smooth curve in M, or a cone over such a submanifold, or a cylinder over such a cone (in a Euclidean space).

For M a general symmetric space none of the above is true except 2 (Lemma 1).

Example 1. Let F^2 be a smooth surface with $\nu = 1$ in a symmetric space $M^3 = M^2(c) \times \mathbb{R}$, which is a graph of a smooth function f over a domain in a 2-space $M^2(c)$ of constant curvature $c \neq 0$. The condition $\nu = 1$ reads as the Monge-Ampere equation

$$\mathrm{Hess}(f) = 0,$$

where Hess is computed with respect to covariant differentiation on $M^2(c)$. One can check that the 1-dimensional distribution L is geodesic if and only if the Gauss curvature of F is constant. By the Gauss equation this means that $\|\nabla f\|$ is constant. For analytic f by the Cauchy-Kowalewski theorem the solution of $\mathrm{Hess}(f) = 0$ depends on two functions of one variable, while the solution of $\|\nabla f\| = const$ depends on only one function of one variable (local solutions of $\|\nabla f\| = const$ are of the form $f = C_1 d + C_2$, where C_1, C_2 are constants and d is the distance function from a point or a curve on $M^2(c)$). So there exist surfaces in $M^2(c) \times \mathbb{R}$ with $\nu = 1$ satisfying neither of the above conditions 1,3 and 4.

However, this is essentially the only case when the behaviour of a submanifold of maximal nullity in a symmetric space differs from that in a space of a constant curvature.

Example 2. Let a symmetric space $M = G/H$ be a left coset space of the Lie group G which is the connected component of the group of isometries of M containing the unit element e. Denote $\pi : G \to G/H = M$ the projection on the corresponding coset and

$L_g : G \to G$ the left multiplication by $g \in G$. Let $\pi(e) = O \in M$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g} = T_e G$, where $\mathfrak{h} = T_e H$ and $\mathfrak{m} = T_O M$.

Now let $M^k(c) \subset M$ be a totally geodesic submanifold of constant curvature $c \neq 0$ in M passing through O. Then for any three vectors e_1, e_2, e_3 in $T_O M^k(c)$ one has

$$[[e_1, e_2], e_3] = (-c)(\langle e_2, e_3\rangle e_1 - \langle e_1, e_3\rangle e_2).$$

Pick a smooth vector function $\tau = \tau(t) \neq 0$ in $\mathfrak{m}$ such that $\tau \perp T_O M^k(c)$ and

$$[[\tau, e_1], e_2] = -c \langle e_1, e_2\rangle \tau \quad \forall e_1, e_2 \in T_O M^k(c)$$

and an arbitrary smooth vector function $E = E(t)$ in $T_O M^k(c)$. Let $\Gamma = \Gamma(t) \subset G$ be a smooth curve which is a solution of the ordinary differential equation

$$d\Gamma/dt = dL_\Gamma(\tau + [\tau, E]).$$

Definition 2. In the above conditions the submanifold

$$F^{k+1} = \Gamma(t)(M^k(c)) \subset M$$

is called *developable.*

A developable submanifold F^{k+1} can be checked to have nullity k, it is of the same constant curvature c and is foliated by totally geodesic leaves congruent to $M^k(c)$ (so it satisfies the conditions 1, 2 and 3), but in general is not contained in any larger totally geodesic submanifold $M^N(c) \subset M$ of constant curvature c.

Definition 3. A submanifold F^n in a symmetric space M is called a (local) *cylinder* over a submanifold $F^k \subset M \quad (1 \leq k \leq n)$ if there exists a totally geodesic submanifold in M locally isometric to a product $M_1 \times M_2$ with $\dim M_2 = n - k$, such that F^k is a submanifold in M_1, and F^n is locally a product of F^k and a domain of M_2. The submanifold M_2 is called *the generatrix* of the cylinder.

The main result of this paper is the following theorem:

Theorem. *Let F^n be a smooth submanifold in a symmetric space M with relative nullity index $\nu(F) = n - 1$ and without planar points. Then either*

1. F^n is locally a cylinder over a surface F^2 of nullity 1 in a totally geodesic submanifold $N^3 \subset M$ locally isometric to a product $M^2(c) \times \mathbb{R}, \quad c \neq 0$; or

2. F^n is locally a cylinder over either

- (a) *a developable submanifold F^k, or*
- (b) *a submanifold F^k of nullity $k-1$ in a totally geodesic flat submanifold $M' \subset M$, or*
- (c) *a smooth curve.*

Remark 1. We assume all the manifolds and submanifolds smooth (C^∞) and the space M simply connected Riemannian globally symmetric.

Remark 2. We do not exclude the case when the generatrix of a cylinder is a point, that is a cylinder over a submanifold is the submanifold itself.

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2. Basic Lemmas

Let $F^n \subset M^{n+p}$ be a submanifold in a Riemannian space. Denote $\langle\, , \rangle$ the inner product of vectors tangent to M and $\tilde{\nabla}$ the Levi-Civita connection on M. Let ∇ be the induced connection on F and $\tilde{R}$ and R the curvature tensors on M and on F respectively. Let $x \in F$ be a point on F. For vector fields X, Y tangent to F and η normal to F defined in a neighbourhood of x one has the following Gauss-Weingarten decompositions:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X \eta = -A_\eta X + D_X \eta,$$

where h is the second fundamental form of $F \subset M$, A_η is the Weingarten operator with respect to the normal vector η acting in the tangent space T_xF, and D is the covariant derivative with respect to the normal connection. Obviously, $\langle A_\eta X, Y\rangle = \langle h(X,Y), \eta\rangle$.

Let L_x be the maximal subspace in T_xF such that $A_\eta L_x = 0$ for any normal vector $\eta \in N_xF$, or equivalently $h(L_x, T_xF) = 0$. Then $\nu(x) = \dim L_x$. If $\nu(x) = n$, then h vanishes at x and we call such a point planar. If this holds in a neighbourhood of x, then the submanifold F is locally totally geodesic in M. We concentrate on the case $\nu(x) \equiv n-1$ on F. Let X be a unit tangent vector field orthogonal to the distribution $L = \cup_x L_x$. Then for $Y, Z \in L$ one has:

$$h(Y,Z) = h(Y,X) = 0, \;\; h(X,X) = a\xi \tag{1}$$

for some nonvanishing smooth function a on F and unit normal vector field ξ (writing $Y \in L$ we mean that Y is a section of L).

In the case when the ambient space is of constant curvature all the submanifolds with $\nu = n-1$ are classified.

Recall, that a diffeomorphic map $f : M_1 \to M_2$ between Riemannian spaces is called *geodesic* if it sends geodesics onto geodesics (in general, without preseving an affine parameter). A geodesic map respects the relative nullity index of submanifolds. Indeed, by the Bianchi theorem the map f is geodesic iff there exists a 1-form ω on M_1 such that for any two vector fields X, Y tangent to M_1 the following condition holds: $df^{-1}(\nabla^2_{df(X)} df(Y)) - \nabla^1_X Y = \omega(X)Y + \omega(Y)X$, where ∇^i is an affine connection on M_i $(i = 1,2)$ [4]. The condition of $(n-1)$-nullity can be expressed in affine terms as follows: $F^n \subset M^{n+p}$ is of the relative nullity $n-1$ if there exists an $(n-1)$-dimensional distribution $L \subset TF$ such that for any two vector fields $Y \in L$ and $T \in TF$ $\;\tilde{\nabla}_Y T$ is still in TF. This is clearly preserved by the map f. Since all the spaces of constant curvature are locally geodesically equivalent it is sufficient to have the classification of submanifolds of relative nullity $n-1$ only when the ambient space is Eulidean. For a curve $\gamma \subset \mathbb{R}^N$ with the first $m-1$ curvatures nonvanishing *the m-th osculating space* at a point $x \in \gamma$ is the subspace in $\mathbb{R}^N$ spanned by the first m vectors of the Frenet frame at x.

Theorem A. [1,3] *Let F^n be a smooth submanifold of nullity $\nu(F) = n-1$ without planar points in a Euclidean space $\mathbb{R}^{n+p}$. Then the submanifold F^n is locally a* *cylinder over either a union of osculating spaces of a smooth curve with nonvanishing corresponding curvatures or a cone over such a submanifold.*

Now let the ambient space be a symmetric space $M = G/H$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the Cartan decomposition of the Lie algebra $\mathfrak{g}$. Then the curvature tensor $\tilde{R}$ of M satisfies the relations

$$\tilde{\nabla}\tilde{R} = 0, \qquad \tilde{R}(A,B)C = -[[A,B],C]$$

for any three vectors A, B, C tangent to M with respect to the Lie brackets in $\mathfrak{g}$ [5]. We will use the implication

$$\tilde{R}(A,C)C = 0 \implies \tilde{R}(A,C) = 0 \tag{2}$$

for vectors A, C tangent to M. Indeed, the Cartan-Killing form B of $\mathfrak{g}$ is strictly negative on the compact subalgebra $\mathfrak{h}$ and $B([A,C],[A,C]) = B(\tilde{R}(A,C)C, A) = 0$, therefore $[A,C] = 0$.

Also recall that by the Cartan theorem a submanifold in a symmetric space is totally geodesic if and only if it is the exponent of a Lie triple system, that is of a subspace $\mathfrak{s} \subset \mathfrak{m}$ such that $[[\mathfrak{s},\mathfrak{s}],\mathfrak{s}] \subset \mathfrak{s}$.

Lemma 1. *Let F^n be a smooth submanifold in a symmetric space M with relative nullity index $\nu(F) = n-1$ and without planar points. Then the distribution L is integrable and one of the following holds:*

1. there exists a vector field $Y \in L$ such that for any vector fields Z, W in the $(n-2)$-dimensional distribution $L' = L \cap Y^\perp$

$$\nabla_Z W \in L, \quad \nabla_Z Y \in L, \quad \nabla_Y Z \in L;$$

2. L is totally geodesic both in F and M.

Proof of the Lemma 1. By the Frobenius theorem to prove integrability of L it is sufficient to show that for any two vector fields $Y, Z \in L$ their Lie bracket (as vector fields, not in the sence of $\mathfrak{g}$) belongs to L, or equivalently

$$\langle \nabla_Z Y - \nabla_Y Z, X \rangle \equiv 0.$$

By the Codazzi equation for the submanifold F one has:

$$\tilde{R}(Y,Z)W^{\perp} = 0$$
$$\tilde{R}(X,Y)Z^{\perp} = \langle \nabla_Y Z, X \rangle \, a \, \xi$$

where $Y, Z, W \in L$ and $\perp$ means the normal component. Let U be another vector field in L. Differentiating the first one of the above equations by U and taking into account the second one and (1) one gets:

$$\langle \nabla_U Y, X \rangle \langle \nabla_Z W, X \rangle - \langle \nabla_U Z, X \rangle \langle \nabla_Y W, X \rangle + \langle \nabla_U W, X \rangle \langle \nabla_Z Y - \nabla_Y Z, X \rangle = 0.$$

Since $\|X\| = 1$ and W is an arbitrary vector field in L, this implies

$$(3) \qquad \langle \nabla_U Y, X\rangle \nabla_Z X - \langle \nabla_U Z, X\rangle \nabla_Y X + \langle \nabla_Z Y - \nabla_Y Z, X\rangle \nabla_U X = 0.$$

Substituting $U = Y$ and taking the inner product with Y one obtains

$$2\,\langle \nabla_Y Y, X\rangle\,\langle \nabla_Z Y - \nabla_Y Z, X\rangle = 0.$$

If L is not integrable, then $\nabla_Z Y - \nabla_Y Z \notin L$ and therefore $\langle \nabla_Y Y, X\rangle = 0$ for any vector field $Y \in L$, which implies $\langle \nabla_Z Y + \nabla_Y Z, X\rangle = 0$ for any two vector fields $Y, Z \in L$. Then substituting $U = Y$ to (3) one has $\langle \nabla_Y Z, X\rangle \nabla_Y X = 0$, that is $\langle \nabla_Y Z, X\rangle = 0$. Therefore L is an integrable distribution.

Now one can rewrite (3) as follows:

$$\langle \nabla_U Y, X\rangle \nabla_Z X - \langle \nabla_U Z, X\rangle \nabla_Y X = 0,$$

that is either $\nabla_L L \subset L$, or for any two vector fields $Y, Z \in L$ the vector fields $\nabla_Z X$ and $\nabla_Y X$ are parallel. In the first case by (1) we have that the leaves of the integral foliation $\mathcal{L}$ of the distribution L are totally geodesic both in F and M. In the second case either $\nabla_L X \equiv 0$, which brings us back to $\nabla_L L \subset L$ and totally geodesic leaves, or $\dim(\nabla_L X) = 1$. Pick a vector field Y such that for any $Z \in L$ orthogonal to Y $\nabla_Z X = 0$ and let L' be a subdistribution in L orthogonal to Y. Since $\nabla_{L'} X = 0$, one has $\nabla_{L'} L \perp X$ which completes the proof of the Lemma. $\square$

Two cases in the Lemma 1 actually lead to the two cases in the Theorem.

We will also need the following Lemma 2 to verify if a submanifold in a symmetric space allows a reduction of codimension. It says that in a vicinity of a generic point x a submanifold F lies in a totally geodesic submanifold whose tangent space at x contains all the covariant derivatives of all orders with respect to F (see [6] for submanifolds of spaces of constant curvature).

Let $x \in F \subset M$ be a point on a smooth submanifold in a Riemannian space. For a smooth curve γ in F passing through x define its *order* $m_x(\gamma)$ at the point x to be the cardinality of its Frenet frame at x viewed as a curve in M; in other words, at the point x the $m_x(\gamma)$-th curvature of $\gamma \subset M$ vanishes while all the i-th curvatures for $i < m_x(\gamma)$ are nonzero. Define *the order* $m(x)$ of the point x to be the maximum of $m_x(\gamma)$ among all smooth curves $\gamma \subset F$ passing through x. We call a point $x \in F$ *generic* if its order is locally maximal. By continuity arguments $m(y) = m(x) = m$ for all the points $y \in F$ sufficiently close to a generic point x. Moreover, the m-th germs of smooth curves with $m_x(\gamma) = m$ form an open dense set in the space of m-th germs of the smooth curves in F passing through x. Therefore we can join the point x to almost all the points in a small neighbourhood of x in F with a smooth curve γ of constant maximal order m. For the point $x \in F$ define succesively the expanding sequence of subspaces $\{N^i\}$ in $N_x F$ in the following way: $N^1 = \operatorname{Span}\{h(X,Y) \bigm| X, Y \in T_x F\}$, $\quad N^{i+1} = \operatorname{Span}\{D_X N^i \bigm| X \in T_x F\}$, where h is the second fundamental form and D is the derivative in the normal connection. Let $N'_x F = N^k$ for k the lowest index such that $N^{k+1} = N^k$. Then all the vectors of Frenet frames of smooth curves in F at the point x belong to $N'(x)$.

Lemma 2. *Let F be a smooth submanifold in a Riemannian space M and x be a generic point in F. Then the submanifold F locally lies in every totally geodesic submanifold $M' \subset M$ passing through x with $T_xM' \supset T_xF \oplus N'_xF$.*

Proof of the Lemma 2. Let $y \in F$ be a point in a neighbourhood of x which can be joined with x by a smooth curve $\gamma \in F$ of constant order m. Pick an arclength parameter s on γ and denote $\tau = \frac{d\gamma}{ds}, \xi_2, \dots, \xi_m$ its Frenet frame as a curve in M. From the above $\{\tau, \xi_2, \dots, \xi_m\}_{|x} \subset T_xF \oplus N'_xF \subset T_xM'$. Construct another curve γ' lieing in M', passing through x, having the same curvature functions as γ, and such that its Frenet frame at x coincides with that of γ. Then locally γ and γ' are the same curve. Indeed, since M' is totally geodesic in M, its second fundamental form vanishes identically and therefore $\nabla'_{\tau'}\xi'_i = \tilde{\nabla}_{\tau'}\xi'_i$, where $\tilde{\nabla}$ and ∇' are the covariant derivatives in M and M' respectively, and $\{\tau', \xi'_2, \dots, \xi'_m\}$ is the Frenet frame for γ'. So the curves γ and γ' satisfy the same Frenet equations in M with the same initial data. By the uniqueness theorem for ODE they must locally coincide. Since y can be chosen almost everywhere in the neighbourhood of x, we are done. $\square$

3. Proof of the Theorem

We will deal separately with two cases in the Lemma 1 which correspond to the cases in the Theorem.

Case 1.

Let F^n be a smooth submanifold in a symmetric space M with relative nullity index $\nu(F) = n-1$ and let x be a generic nonplanar point on F. Without loss of generality let $x = \pi(e)$, where $\pi : G \to G/H = M$; then $\mathfrak{m} = T_xM$ is a Lie triple system in the Lie algebra $\mathfrak{g} = T_eG$.

According to the first case in the Lemma 1 in the neighbourhood of x there exists a vector field $Y \in L$ such that for the $(n-2)$-dimensional distribution $L' = L \cap Y^\perp$ both $\nabla_{L'}L \perp X$ and $\nabla_L L' \perp X$, but $\nabla_Y Y \notin L$. Choose Y to be a unit vector field. By (1) one has:

$$\tilde{\nabla}_{L'}L \subset L, \quad \tilde{\nabla}_L L' \subset L, \quad \tilde{\nabla}_Y Y = bX + W, \tag{4}$$

where $b = \langle \nabla_Y Y, X\rangle \neq 0$ and W is a vector field in L'.

The equations of Gauss-Codazzi for the submanifold F read as follows:

$$\begin{aligned}
\tilde{R}(T_1, T_2, T_3, T_4) &= R(T_1, T_2, T_3, T_4) \\
\tilde{R}(L, L)L^{\perp} &= 0 \\
\tilde{R}(X, Y)L'^{\perp} &= \tilde{R}(X, L')Y^{\perp} = 0 \\
\tilde{R}(X, Y)Y^{\perp} &= a\, b\, \xi \\
\tilde{R}(U, X)X^{\perp} &= a\, D_U\xi + (U(a) - a\, \langle \nabla_X X, U\rangle)\, \xi \\
\tilde{R}(Y, X)X^{\perp} &= a\, D_Y\xi + (Y(a) - a\, \langle \nabla_X X, Y\rangle)\, \xi,
\end{aligned} \tag{5}$$

where T_i are any vector fields tangent to F, U is a vector field in L', and $^\perp$ means the normal component.

The proof is divided into a series of propositions.

Proposition 1. *The minimal Lie triple system in T_xM containing T_xF and all the successive covariant derivatives of all orders of vector fields tangent to F by vector fields tangent to F coincides with that containing T_xF.*

Proof of Proposition 1. It is sufficient to show that the vector ξ and all its derivatives of all orders in the normal connection at x are linear combinations of vectors from T_xF and the vectors that can be obtained from them by successive actions of $\tilde{R}$. By (5)

$$\xi = (ab)^{-1}\tilde{R}(X,Y)Y + \{\text{tangent vectors}\}.$$

Suppose all the derivatives of ξ in the normal connection up to order k are already expressed as linear combinations of tangent vectors and vectors that can be obtained from them by successive actions of $\tilde{R}$. Differentiating the $(k+1)$-th time and taking into account that $\tilde{\nabla}\tilde{R} = 0$ we get the desired expression substituting ξ from the above. $\square$

Therefore when applying Lemma 2 it is sufficient to take a Lie triple system spanned by T_xF.

Proposition 2. *L' is a Lie triple system in T_xM and the following holds:*

$$\nabla_{L'}X = \nabla_{L'}Y = 0.$$

Proof of Proposition 2. By (4) for any vector field U in L' the vector field $\tilde{\nabla}_U X$ is tangent to F and is orthogonal to L. Since X is the unit vector we have $\tilde{\nabla}_U X = 0$.

By (5) $\tilde{R}(X,Y)V^\perp = 0$ for any vector field $V \in L'$. Differentiation this by a vector field $U \in L'$ and applying (4), (5) and $\tilde{\nabla}_U X = 0$ we arrive at the equation $\langle \nabla_U V, Y\rangle\, a\, b\, \xi = 0$ which gives $\tilde{\nabla}_{L'}L' \subset L'$. So the distribution L' is integrable and its integral leaves are totally geodesic. In particular, L'_x is a Lie triple system. Rewriting the same as $\tilde{\nabla}_{L'}Y \perp L'$ we deduce that $\tilde{\nabla}_{L'}Y = 0$. $\square$

Proposition 3. *For $U, V \in L'$*

$$\tilde{R}(X,U)V = R(X,U,V,X)\,X, \quad \tilde{R}(Y,U)V = R(Y,U,V,Y)\,Y$$

Proof of Proposition 3. By (5) $\tilde{R}(X,U)V$ has no normal components. Moreover, by Proposition 2 $\tilde{R}(Z,V,U,X) = 0$ for any $Z \in L'$ since L' is a Lie triple system. Therefore $\tilde{R}(X,U)V \perp L'$. Finally, $\langle \tilde{R}(X,U)V, Y\rangle = \tilde{R}(Y,V,U,X) = 0$ by (4) and the definition of the curvature tensor. Thus $\tilde{R}(X,U)V \parallel X$.

In the same way, the vector $\tilde{R}(Y,U)V$ has neither normal, nor L'-components and is orthogonal to X, that is $\tilde{R}(Y,U)V \parallel Y$. Applying the Gauss equation from (5) one obtains the corresponding coefficients in the right-hand sides. $\square$

Proposition 4. *For $U \in L'$ $\quad R(X,Y,Y,U)=0$*

Proof of Proposition 4. Rewriting the equations from the previous Proposition in terms of the Lie brackets one obtains, in particular, that $[[Y,U],U] \parallel Y$. So $[[[Y,U],U],Y]=0$ in the Lie algebra $\mathfrak{g}$. By the Jacobi identity this gives $[[[Y,U],Y],U] = 0$ or $[[[[Y,U],Y],U],A]=0$ for any vector $A \in \mathfrak{m} = T_xM$. Back to the curvature tensor, this is equivalent to

$$\tilde{R}(\tilde{R}(Y,U)Y,U)A=0.$$

By (5) $\tilde{R}(Y,U)Y$ has no normal components, so $\tilde{R}(Y,U)Y = R(Y,U,Y,X)X+Z$, where $Z \in L'$. Taking $A=X$ and noticing that $\tilde{R}(Z,U)X = 0$ by Proposition 2, we get $R(Y,U,Y,X)\tilde{R}(X,U)X = 0$. If $R(Y,U,Y,X) \neq 0$ for at least one vector $U \in L'$, then $\tilde{R}(X,U)X \equiv 0$. By the same arguments substituting $A=Y$ we obtain $\tilde{R}(X,U)Y \equiv 0$. Differentiating this by Y and taking the normal component we get $\langle \nabla_Y U, Y\rangle = 0$. Differentiating $\tilde{R}(X,U)X = 0$ by Y we arrive at $\tilde{R}(Y,U)X = 0$ and therefore $R(X,Y,Y,U)=0$. $\square$

Proposition 5. *For $U,V \in L'$*

$$\tilde{R}(X,U)V = \tilde{R}(Y,U)V = 0$$

Proof of Proposition 5. By (5) $\tilde{R}(U,Y,Y,\eta)=0$ for any normal vector field η. Differentiating this by X one obtains $\langle \nabla_X U, X\rangle = 0$, that is $\tilde{\nabla}_X L' \subset L$. Hence $R(X,U,V,X) = \langle \nabla_X\nabla_U V - \nabla_U\nabla_X V - \nabla_{\nabla_X U - \nabla_U X} V, X\rangle = 0$ by (4) and Proposition 2. It follows from Proposition 3 that $\tilde{R}(X,U)V=0$. Differentiating this by Y and then taking the inner product with Y, one gets $R(Y,U,V,Y)=0$ and therefore $\tilde{R}(Y,U)V=0$. $\square$

Proposition 6. *Let $\mathfrak{r}$ be the minimal Lie triple system in $\mathfrak{m} = T_xM$ containing the vectors X and Y and $\mathfrak{s}$ be the minimal Lie triple system in $\mathfrak{m}$ containing T_xF. Then $\mathfrak{s} = \mathfrak{r} \oplus L'_x$ and $[\mathfrak{r}, L'_x] = 0, \quad \langle \mathfrak{r}, L'_x\rangle = 0$.*

Proof of Proposition 6. From Proposition 5 $[[X,U],U] = [[Y,U],U] = 0$ for any vector $U \in L'_x$. Then by (2) $\quad [X,U]=[Y,U]=0$. Therefore $[\mathfrak{r}, L'_x]=0$ since $\mathfrak{r}$ is generated by X and Y. The equation now $\langle \mathfrak{r}, L'_x\rangle = 0$ follows inductively from $\langle L'_x, X\rangle = \langle L'_x, Y\rangle = 0$. Hence $\mathfrak{r} \oplus L'$ is a Lie triple system in $\mathfrak{m}$ and it coincides with $\mathfrak{s}$. $\square$

Applying Lemma 2 and Proposition 1 we obtain that locally the submanifold F^n is a cylinder with an $(n-2)$-dimensional generatrix $M_2 = \operatorname{Exp} L'_x$ over a surface $F^2 \in N = \operatorname{Exp} \mathfrak{r}$ such that $T_xF^2 = \operatorname{Span}\{X,Y\}$. Since the covariant derivatives in N and in M are the same for vector fields tangent to N one can easily see that the surface $F^2 \subset N$ has nullity $\nu = 1$ and all the relations involving X and Y from (4) and (5) still hold.

To complete the proof we have to show that N is locally isometric to a product of a line and a plane of constant curvature $M^2(c)$. In fact, it is sufficient to show that $\dim N = 3$. Indeed, all the symmetric spaces of dimension 3 are either of constant curvature or locally isometric to $M^2(c) \times \mathbb{R}$. But the case of constant curvature is not possible since otherwise $\tilde{R}(X,Y,Y,\xi) = 0$ that contradicts (5). So it is sufficient to prove the following Proposition:

Proposition 7. *The space* $\mathrm{Span}\,\{X, Y, \xi\}$ *is a Lie triple system in* $\mathfrak{m}$*, that is* $\mathfrak{x} = T_xN = \mathrm{Span}\,\{X, Y, \xi\}$.

Proof of Proposition 7. First show that $D_Y\xi = 0$. Indeed, by (5) $\tilde{R}(X, Y)Y = KX + ab\xi$, where K is the Gauss curvature of F^2. Differentiating this by Y and applying (4) and $\tilde{R}(Y, X)X^{\perp} = aD_Y\xi + (Y(a) - a\langle\nabla_X X, Y\rangle)\xi$, one obtains

$$2abD_Y\xi + (Y(ab) + bY(a) - ab\langle\nabla_X X, Y\rangle)\xi + Y(K)X = 0.$$

Therefore $D_Y\xi = 0$ and $Y(K) = 0$. Choose orthonormal vector fields X_1 and X_2 tangent to F^2 such that $\tilde{R}(X_2, X_1)X_1^{\perp} = 0$, that is

$$\begin{aligned} \tilde{R}(X_2, X_1)X_1 &= KX_2 \\ \tilde{R}(X_1, X_2)X_2 &= KX_1 + f\xi, \end{aligned} \tag{6}$$

where f is a smooth nonvanishing function in a neighbourhood of x. Rewrite the second one of the equations (6) as $[[X_1, X_2], X_2] = -KX_1 - f\xi$ and take its Lie bracket with X_1. Then by the Jacobi identity and (6)

$$[\xi, X_1] = 0. \tag{7}$$

In particular, $\tilde{R}(X_1, X_2)\xi = \tilde{R}(\xi, X_2)X_1$.

Differentiating the first equation of (6) by X and applying (4), (6) and (7) we get

$$a\langle X, X_1\rangle\tilde{R}(X_2, X_1)\xi = X(K)X_2 + (f\langle\nabla_X X_1, X_2\rangle + Ka\langle X, X_2\rangle)\xi.$$

Hence $\tilde{R}(X_2, X_1)\xi \in \mathrm{Span}\,\{X_1, X_2, \xi\} = \mathrm{Span}\,\{X, Y, \xi\}$. Since $\tilde{R}(X_2, X_1)\xi$ is orthogonal to ξ and to X_1 (by (7)) one can see from (6) that $\tilde{R}(X_2, X_1)\xi = -fX_2$. Taking the Lie bracket of $[[X_2, X_1], \xi] = fX_2$ with X_1 one gets $f[X_1, X_2] = K[\xi, X_2]$. If $K = 0$ then $[X_1, X_2] = 0$ and it follows from (6) that $f = 0$. So $K \neq 0$ and

$$[\xi, X_2] = (f/K)[X_1, X_2],$$

which proves the Proposition together with (6) and (7). □

This completes the proof of the first case of the Theorem.

As the Example 1 shows, there exist surfaces in $M^2(c) \times \mathbb{R}$ with nullity $\nu = 1$ that are not foliated by geodesics (so that $b = \langle\nabla_Y Y, X\rangle \neq 0$).

Case 2.

Let F^n be a smooth submanifold in a symmetric space M with relative nullity index $\nu(F) = n - 1$ and let x be a generic nonplanar point on F. According to the second case in Lemma 1 in the neighbourhood of x the $(n-1)$-dimensional distribution L is integrable and its integral foliation $\mathcal{L}$ is totally geodesic both in F and M. By (1) one has:

$$\tilde{\nabla}_L L \subset L, \quad \tilde{\nabla}_L X = 0 \tag{8}$$

The equations of Gauss-Codazzi for the submanifold F read as follows:

$$
\begin{aligned}
\tilde{R}(T_1, T_2, T_3, T_4) &= R(T_1, T_2, T_3, T_4) \\
\tilde{R}(L, L)L^{\perp} &= 0 \\
\tilde{R}(X, L)L^{\perp} &= 0 \\
\tilde{R}(U, X)X^{\perp} &= aD_U\xi + (U(a) - a\langle \nabla_X X, U\rangle)\xi,
\end{aligned} \tag{9}
$$

where T_i are any vectors fields tangent to F, and U is a vector field in L.

We split the proof into a series of propositions. First study the intrinsic geometry of F.

Proposition 8. *The submanifold F^n is a locally symmetric space. It is locally isometric to a Riemannian product $F^k(c) \times M'$, where $F^k(c)$ is a space of constant curvature c and M' is a locally symmetric space of dimension $n-k$. The leaves of the foliation $\mathcal{L}$ are the products of hyperplanes in $F^k(c)$ and M'.*

Proof of Proposition 8. First prove that F is locally symmetric. Differentiating the Gauss equation (the first one in (9)) by any vector field T_5 tangent to F and applying (9) and (8) again, we obtain $(\nabla_{T_5} R)(T_1, T_2, T_3, T_4) = 0$, that is $\nabla R \equiv 0$. Notice that although F is locally symmetric in a symmetric space M, it is not in general totally geodesic.

Each leaf of $\mathcal{L}$ is totally geodesic in F. So we have a locally symmetric space foliated by a family of totally geodesic hypersurfaces. If F is irreducible, then by [7] the codimension of a totally geodesic submanifold is greater than or equal to the rank. So F must be of rank one. All the totally geodesic submanifolds in ROSS's are classified and one can see that only spaces of constant curvature admit totally geodesic hypersurfaces [7].

Let F be reducible and $F = F_0 \times F_1 \times \cdots \times F_m$ be its de Rham decomposition with a Euclidean factor F_0 and irreducible non-Euclidean symmetric factors F_α $(\alpha \geq 1)$. Then $\mathfrak{f} = T_x F$ splits in an orthogonal sum of Lie triple systems $\mathfrak{f}_\alpha = T_x F_\alpha$ $(0 \leq \alpha \leq m)$ with respect to the Lie triple brackets in $\mathfrak{f}$. Let X_α be the projection of the vector X on $\mathfrak{f}_\alpha$. If $X_\alpha \neq 0$ then its orthogonal complement in $\mathfrak{f}_\alpha$ is a subspace $L_x \cap \mathfrak{f}_\alpha$ and is a Lie triple system itself. So F_α admits a totally geodesic hypersurface and therefore is of constant curvature. Moreover, any totally geodesic hypersurface in a product $F_\alpha(c_\alpha) \times F_\beta(c_\beta)$ of two spaces of constant curvatures is a product of one of them and a hyperplane in the second one. Therefore all but one of the X_α's $(0 \leq \alpha \leq m)$ are zero and the only nonvanishing component is tangent to a factor F_α of constant curvature. Denote this factor $F^k(c)$ and the product of all the others M'. Then the integral submanifolds of L are exactly the products of hyperplanes in $F^k(c)$ and the whole M'. $\square$

We are going to show that not only F^n splits as a Riemannian manifold but also as a submanifold in M.

Proposition 9. *Let $N \subset M$ be the minimal (by inclusion) totally geodesic submanifold locally containing $F^k(c)$. Then the submanifold $F^k(c) \subset N$ is of nullity $k-1$ and F^n is a cylinder over $F^k(c)$ with the generatrix M'.*

Proof of Proposition 9. The subspace $\mathfrak{m}' = T_xM'$ is a tangent space to the totally geodesic submanifold and is therefore a Lie triple system in $\mathfrak{m} = T_xM$. The Lie triple system $\mathfrak{n} = T_xN$ is the minimal Lie triple system in $\mathfrak{m}$ containing the tangent space $T_xF^k(c)$ and all the covariant derivatives of all orders of tangent vector fields to $F^k(c)$ along $F^k(c)$ at the point x.

For a vector field V tangent to M' the normal component of the covariant derivative $\tilde{\nabla}_X V$ vanishes due to (1), while the tangent component is in $\mathfrak{m}'$ by Proposition 8. Moreover, it follows from (9) and Proposition 8 that $\tilde{R}(X,V)V = 0$. By (2) this implies $[X,V] = 0$. In the same way for any $Y \perp X$ tangent to $F^k(c)$ $[Y,V] = 0$. Hence the fields of linear operators $\tilde{R}(X,V)$ and $\tilde{R}(Y,V)$ vanish identically:

$$\tilde{R}(X,V) = \tilde{R}(Y,V) \equiv 0.$$

In particular, by (9) one immediately gets $D_V\xi = 0$.

We show that the minimal Lie triple system spanned by $\mathfrak{n}+\mathfrak{m}'$ is in fact the minimal Lie triple system containing T_xF^n and all the covariant derivatives of all orders of tangent vector fields to F^n along F^n at the point x. It is sufficient to prove that all the covariant derivatives in the normal connection of the form $D_V D_{X_1} \dots D_{X_m}\xi$ can be expressed as linear combinations of covariant derivatives of the form $D_{X_1} \dots D_{X_q}\xi$, where V is a vector field tangent to M' and $\{X_i\}$ are vector fields tangent to $F^k(c)$. For $m = 0$ it is already known. For $m > 0$ by the above equation $\tilde{R}(V,X_1)D^{m-1}\xi = 0$, where $D^{m-1}\xi = D_{X_2} \dots D_{X_m}\xi$. Then the Ricci equation implies that $D_V D_{X_1} D^{m-1}\xi = D_{X_1} D_V D^{m-1}\xi + D_{\nabla_V X - \nabla_X V} D^{m-1}\xi$ and we are done by induction.

To finish the proof it is sufficient to show that $[\mathfrak{n},\mathfrak{m}'] = 0$ in the Lie algebra $\mathfrak{g}$ (in particular, this implies $\langle \mathfrak{n},\mathfrak{m}' \rangle = 0$). First notice that since F^n is a Riemannian product, $\nabla_X V, \nabla_Y V \in \mathfrak{m}'$ and therefore $\tilde{\nabla}_X V, \tilde{\nabla}_Y V \in \mathfrak{m}'$ for any vector fields V tangent to M' and $Y \perp X$ tangent to $F^k(c)$. Differentiating $\tilde{R}(X,V) = 0$ by X one obtains $\tilde{R}(\xi,V) = 0$. Differentiating this successively by any vector fields $X_m, \dots, X_1$ tangent to $F^k(c)$ one arrives at $\tilde{R}(D^m\xi,V) = 0$, where $D^m\xi = D_{X_1} \dots D_{X_m}\xi$. So the Lie brackets of the vector V with all the vectors of $T_xF^k(c), \xi$ and all the $D^m\xi$ vanish. Now it can be easily seen by induction that the Lie brackets of V with the minimal Lie subalgebra $\mathfrak{u} \subset \mathfrak{g}$ containing all these vectors also vanish. Since $\mathfrak{n} = \mathfrak{u} \cap \mathfrak{m}$ and since V is an arbitrary vector in $\mathfrak{m}'$, we are done. $\square$

Now we concentrate on the structure of the submanifold $F^k(c) \subset N$. It is of constant curvature c and is of nullity $k-1$. The equations (8) and (9) still hold for $F^k(c)$. Let $\mathfrak{k} = \mathfrak{u} \cap \mathfrak{h}$. Then $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{n}$ is the Cartan decomposition of the Lie subalgebra $\mathfrak{u} \subset \mathfrak{g}$ and the symmetric space N is the left coset space U/K, where $U \subset G$ and $K \subset H$ are the Lie groups of $\mathfrak{u}$ and $\mathfrak{k}$ respectively. Let $\pi : U \to N = U/K$ be the projection on the corresponding coset and denote $\pi(e) = O \in N$. For $u \in U$ let L_u and R_u be the left and the right multiplications by u in U and $\mathrm{Ad}_p : \mathfrak{n} \to \mathfrak{n}$ the adjoint action of $p \in K$ on $\mathfrak{n}$, $\mathrm{Ad}_p = dL_p dR_p^{-1}$. Denote also $\exp : \mathfrak{g} \to G$ the exponential map and $\mathrm{Exp} : \mathfrak{n} \to N$ the geodesic exponential map from $\mathfrak{n} = T_O N$ to the Riemannian manifold N. It is known ([5]) that $\pi(\exp_{|\mathfrak{n}}) = \mathrm{Exp}$.

Now let $\gamma = \gamma(t) \subset N$ be an integral curve of the vector field X with t an arclength parameter and $\Gamma = \Gamma(t) \subset U$ its lift to U. Choose an orthonormal frame field $\{E_1, \dots, E_{k-1}\}$ in the bundle $T_{\gamma(t)}F^k \cap X^\perp$ along the curve γ and denote $e_i = d\Gamma^{-1}E_i$ $(1 \le i \le k-1)$, $\quad \tau = d\Gamma^{-1}(\dot\gamma) = d\pi_{|e} dL_\Gamma^{-1}(\dot\Gamma)$ where dot is for $\frac{d}{dt}$ and Γ is viewed as an isometry of N; one has a decomposition $dL_\Gamma^{-1}(\dot\Gamma) = \tau + P$, $\ \tau \in \mathfrak{n}$, $P \in \mathfrak{k}$. The vectors $\{\tau, e_1, \dots, e_{k-1}\} \subset \mathfrak{n}$ are orthonormal. The submanifold $F^k \subset N$ can be locally defined as follows:

$$x(t, u^1, \dots, u^{k-1}) = \Gamma(t)\left(\operatorname{Exp}\sum_{i=1}^{k-1} u^i e_i(t)\right) = \pi\left(\Gamma(t)\exp\sum_{i=1}^{k-1} u^i e_i(t)\right). \tag{10}$$

Denote $A = \sum_{i=1}^{k-1} u^i e_i$ and $\mathcal{X} = \Gamma \exp A \subset U$. Then $x = \pi(\mathcal{X})$.

Since F^k is of constant curvature c and in view of (9) $\tilde R(X, E_i)E_j = c\,\delta_{ij}X$, $\tilde R(E_i, E_j)E_k = c\,(\delta_{jk}E_i - \delta_{ik}E_j)$, that implies

$$[[\tau, e_i], e_j] = -c\,\delta_{ij}\tau, \quad [[e_i, e_j], e_k] = -c\,(\delta_{jk}e_i - \delta_{ik}e_j). \tag{11}$$

Proposition 10. *The submanifold $F^k \subset N$ defined by* (10) *with orthonormal vector functions $\tau, e_1, \dots, e_{k-1}$ respecting* (11) *is of nullity $k-1$ if and only if*

$$\dot e_i + [P, e_i] \in \operatorname{Span}\{\tau, e_1, \dots, e_{k-1}\} \quad \forall i = 1, \dots, k-1.$$

Proof of Proposition 10. In a standard fashion [5] compute the first and the second covariant derivatives of the submanifold (10) at a point $x = x(t,u) = \pi(\mathcal{X}(t,u))$ and translate them to $\mathfrak{n}$ by the linear isometry $d\,(\mathcal{X}(t,u))^{-1} : T_xN \to T_ON$. One has:

$$\begin{aligned} d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial u^i}x) &= d\pi_{|e}dL_{\mathcal{X}}^{-1}\left(\frac{\partial \mathcal{X}}{\partial u^i}\right) = d\pi_{|e}dL_{\mathcal{X}}^{-1}\big(dL_\Gamma \frac{\partial \exp A}{\partial u^i}\big) \\ &= d\pi_{|e}dL_{\mathcal{X}}^{-1}dL_\Gamma dL_{\exp A}\left(\sum_{j=0}^{\infty}\frac{(\operatorname{ad}_{-A})^j e_i}{(j+1)!}\right) = d\pi_{|e}\left(\sum_{j=0}^{\infty}\frac{(\operatorname{ad}_{-A})^j e_i}{(j+1)!}\right), \end{aligned}$$

where $\operatorname{ad}_T : \mathfrak{u} \to \mathfrak{u}$ is the linear operator such that $\operatorname{ad}_T(Q) = [T, Q]$ for $T, Q \in \mathfrak{u}$. In view of (11) $\sum\limits_{j=0}^{\infty}\frac{(\operatorname{ad}_{-A})^j e_i}{(j+1)!} = \frac{\sinh(\chi|u|)}{\chi|u|}e_i - \frac{\sinh(\chi|u|)-\chi|u|}{\chi|u|^3}u^iA + \frac{\cosh(\chi|u|)-1}{\chi^2|u|^2}[e_i, A]$, where $\chi = \sqrt{-c} \in \mathbb{R} \cup i\mathbb{R}$ and $|u| = \sqrt{\sum (u^i)^2}$. Therefore

$$d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial u^i}x) = \frac{\sinh(\chi|u|)}{\chi|u|}e_i - \frac{\sinh(\chi|u|) - \chi|u|}{\chi|u|^3}u^iA$$

and the $\mathfrak{k}$-component of the vector $dL_{\mathcal{X}}^{-1}(\frac{\partial \mathcal{X}}{\partial u^i})$ is $\frac{\cosh(\chi|u|)-1}{\chi^2|u|^2}[e_i, A]$. Notice, that $\operatorname{Span}\{d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial u^1}x), \dots, d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial u^{k-1}}x)\} = \operatorname{Span}\{e_1, \dots, e_{k-1}\}$.

In the similar way

$$d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial t}x) = d\pi_{|e} dL_{\mathcal{X}}^{-1}\left(\frac{\partial\mathcal{X}}{\partial t}\right) = d\pi_{|e} dL_{\mathcal{X}}^{-1}\big(dL_\Gamma \frac{\partial \exp A}{\partial t} + dR_{\exp A}\dot\Gamma\big)$$
$$= d\pi_{|e}\left(\sum_{j=0}^{\infty}\frac{(\mathrm{ad}_{-A})^j \dot A}{(j+1)!} + \mathrm{Ad}_{\exp A}^{-1} dL_\Gamma^{-1}\dot\Gamma\right).$$

Since $\mathrm{Ad}_{\exp} = \exp\mathrm{ad}$ and by (11) we get

$$d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial t}x) = \dot A + [P,A] + \frac{1}{3!}[[\dot A + [P,A],A],A] + \cdots + \cosh(\chi|u|)\tau.$$

Recall, that the covariant derivative of a vector field Y along a smooth curve $\sigma = \sigma(s)$ in the symmetric space N can be computed as follows: pick a smooth lift $\Sigma = \Sigma(s)$ of the curve σ to U and set $T = d\Sigma^{-1}(Y) \in \mathfrak{n}$. Then

$$d\Sigma^{-1}(\tilde\nabla_{d\sigma/ds}Y) = \frac{dT}{ds} + \big[\big(dL_\Sigma^{-1}\frac{d\Sigma}{ds}\big)^{\mathfrak{k}}, T\big],$$

where the superscript $\mathfrak{k}$ is for the $\mathfrak{k}$-component of the corresponding vector.

From the above one can see that the covariant derivative

$$d\mathcal{X}^{-1}\big(\tilde\nabla_{\partial/\partial u^i}\frac{\partial x}{\partial u^j}\big)$$

is a linear combination of $e_1,\dots,e_{k-1}$, or in other words all the covariant derivatives $\tilde\nabla_{\partial/\partial u^i}\tilde\nabla_{\partial/\partial u^j}x$ are tangent to the submanifold F^k. For the second fundamental form of F^k this means that $h(\partial/\partial u^i, \partial/\partial u^j) = 0$ for all $i,j = 1,\dots,k-1$. The $(k-1)$-nullity condition now reads as $h(\partial/\partial u^i, \partial/\partial t) = 0$ for any $i = 1,\dots,k-1$, or equivalently, $\tilde\nabla_{\partial/\partial u^i}\frac{\partial x}{\partial t} \in T_xF^k$. Substituting from the above one has

$$d\mathcal{X}^{-1}\big(\tilde\nabla_{\partial/\partial u^i}\frac{\partial x}{\partial t}\big) = \frac{\partial}{\partial u^i}\big(\dot A + [P,A] + \frac{1}{3!}[[\dot A + [P,A],A],A] + \cdots + \cosh(\chi|u|)\tau\big)$$
$$+\frac{\cosh(\chi|u|)-1}{\chi^2|u|^2}\big[[e_i,A], \dot A + [P,A] + \frac{1}{3!}[[\dot A + [P,A],A],A] + \cdots + \cosh(\chi|u|)\tau\big].$$

In particular, for $u^1 = \cdots = u^{k-1} = 0$

$$d\mathcal{X}^{-1}\big(\tilde\nabla_{\partial/\partial u^i}\frac{\partial x}{\partial t}\big) = \dot e_i + [P,e_i], \quad d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial t}x) = \tau$$

Therefore $d\mathcal{X}_{|u=0}^{-1}(T_xF^k) = \mathrm{Span}\,\{e_1,\dots,e_{k-1},\tau\}$ and the condition of the $(k-1)$-nullity is $\dot e_i + [P,e_i] \in \mathrm{Span}\,\{e_1,\dots,e_{k-1},\tau\}$ for all i.

In fact, these relations give not only the necessary, but also the sufficient conditions for a submanifold (10) to be of nullity $k-1$. Substituting them to the above formulae and applying (11) one gets that

$$d\mathcal{X}^{-1}(\tilde\nabla_{\partial/\partial t}x) \in \mathrm{Span}\,\{e_1,\dots,e_{k-1},\tau\},$$

that is $d\mathcal{X}^{-1}(T_x F^k) = \mathrm{Span}\{e_1, \dots, e_{k-1}, \tau\}$ and $d\mathcal{X}^{-1}(\tilde{\nabla}_{\partial/\partial u^i} \frac{\partial x}{\partial t})$ is a linear combination of the vectors from this linear span. $\square$

At this point we separate the end of the proof for the Euclidean case ($c = 0$) and for the non-Euclidean one. It appears that the flat submanifold F^k of nullity $k-1$ is contained in a flat totally geodesic submanifold in the ambient symmetric space. The analogue of that for $c \neq 0$ is not true, as the following simple example shows:

Example 3. Let $N = \mathbb{R} \times S^2(1)$. Choose the local coordinates $t \in \mathbb{R}$, $r \in (0, \pi)$, $\phi \in [0, 2\pi)$ such that the metric element of N looks as $ds^2 = dt^2 + dr^2 + \sin^2 r d\phi^2$. The surface $t = r$ (the graph of the function r over the sphere $S^2(1)$, compare with the Example 1) is of relative nullity 1, is foliated by geodesics of N and is of the constant curvature $\frac{1}{2}$. It is clearly not contained in any totally geodesic sphere of N of curvature $\frac{1}{2}$, because there are none of them.

On the other hand, in the non-flat case the vectors $\{e_i\}$ can be chosen to be constants. Therefore, by (10) the submanifold $F^k(c)$ is spanned by moving a *fixed* totally geodesic submanifold $F^{k-1}(c)$ by a curve Γ in the isometry group U (notice, that in general there exist isometric totally geodesic submanifolds of constant curvature which are not congruent). One needs only to ensure that Γ satisfies both (11) and the relations of Proposition 10. This gives developable submanifolds:

Proposition 11. *A submanifold $F^k(c) \subset N$ of the nullity $k-1$ with $c \neq 0$ can be constructed as follows: pick orthonormal vectors $\{e_1^0, \dots, e_{k-1}^0\} \in \mathfrak{n}$ spanning a Lie triple system $\mathfrak{e}$ of constant curvature c and a unit vector function $\tau = \tau(t) \subset \mathfrak{n}$ satisfying* (11)*. Let $E = E(t)$ be an arbitrary vector function in $\mathfrak{e}$ and $\Gamma = \Gamma(t)$ be a curve in U such that $dL_\Gamma^{-1}\dot{\Gamma} = \tau + [\tau, E]$. Then the submanifold $F^k(c)$ is defined by*

$$x(t, u^1, \dots, u^{k-1}) = \Gamma(t)\big(\mathrm{Exp} \sum\nolimits_{i=1}^{k-1} u^i e_i^0\big) = \Gamma(t)\big(\mathrm{Exp}\, \mathfrak{e}\big),$$

with Γ acting as an isometry of N.

Proof of Proposition 11. By Proposition 10 the submanifold $F^k(c)$ is of the form (10) and there exist decompositions $\dot{e}_i + [P, e_i] = \sum_{j=1}^{k-1} \alpha_{ij} e_j + \mu_i \tau$ for $1 \leq i \leq k-1$. The $(k-1) \times (k-1)$ matrix (α_{ij}) is skew-symmetric and therefore one can pick another orthonormal basis (depending on t) in $\mathrm{Span}\{e_1, \dots, e_{k-1}\}$ to make this matrix zero. Keeping the same notations for this new basis one has $\dot{e}_i + [P, e_i] = \mu_i \tau$.

Furthermore, we took the curve $\Gamma \subset U$ to be an arbitrary lift of the curve $\gamma \subset N$. Let $V = V(t)$ be a smooth curve in the isotropy subgroup K and $\Gamma' = \Gamma V^{-1}$. Denote $Q = Q(t) = dL_V^{-1}\dot{V} \subset \mathfrak{k}$. Now $\pi(\Gamma')$ is still the curve γ and the new vectors $\{e_i'\}, \tau'$ and P' can be expressed as follows:

$$e_i' = \mathrm{Ad}_V\, e_i, \quad \tau' = \mathrm{Ad}_V\, \tau, \quad P' = \mathrm{Ad}_V(P - Q).$$

The relations (11) are still satisfied for the orthonormal vectors $\{e_1', \dots, e_{k-1}', \tau'\}$ and

$$\dot{e}_i' = (\mathrm{Ad}_V\, e_i)^{\cdot} = \mathrm{Ad}_V(\dot{e}_i + [Q, e_i]) = \mathrm{Ad}_V(\mu_i \tau + [Q - P, e_i]).$$

Set $Q = P + c^{-1}[\tau, \sum_j \mu_j e_j]$ and reconstruct the curve $V(t)$ in the group K solving the corresponding ODE system. Then by (11) $\dot{e}'_i \equiv 0$. Omitting dashes one has $[P, e_i] = \mu_i \tau$ for the constant vectors $\{e_i\}$.

By (11) these equations are satisfied for the vector $P_1 = c^{-1}[\tau, \sum_j \mu_j e_j]$. So the vector P can be decomposed as $P_1 + P_2$, where $P_2 \in \mathfrak{k}$ annulates all the e_i's, that is $[P_2, e_i] \equiv 0$. Constructing a curve $W = W(t)$ in the isotropy group K such that $P_2 = dL_W^{-1}\dot{W}$ and taking $\Gamma' = \Gamma W^{-1}$ instead of Γ one gets the new vectors $e'_i = \mathrm{Ad}_W\, e_i$ which are still constant and which satisfy (11) together with the new vector $\tau' = \mathrm{Ad}_W\, \tau$. The vector $P \in \mathfrak{k}$ transforms to the vector $P' = \mathrm{Ad}_W(P - P_2) = c^{-1}[\tau', \sum_j \mu_j e'_j]$. Denoting $e^0_i = e'_i$ and $E(t) = \sum_{j=1}^{k-1} \mu_j(t) e^0_j$ and again omitting dashes one has $\dot{e}^0_i \equiv 0$ and $dL_\Gamma^{-1}\dot{\Gamma} = \tau + [\tau, E]$.

Notice, that in general we cannot get rid of the vector function E even by reparametrization $u^i = \tilde{u}^i + \phi^i(t)$ for some smooth functions $\{\phi^1, \dots, \phi^{k-1}\}$. For example, it is not possible when $c < 0$ and $\|E\|^2 \geq -c$. $\square$

Now consider the flat case. It occurs that a flat submanifold F^k of nullity $k-1$ in a symmetric space behaves very similar to that in a Euclidean case.

Proposition 12. *A flat submanifold $F^k \subset N$ of nullity $k-1$ is a cylinder with a flat generatrix either over a curve or over a submanifold F^l of nullity $l-1$ in a flat totally geodesic submanifold $N' \subset N$.*

Proof of Proposition 12. In view of Proposition 10 the submanifold $F^k \subset N$ can be defined by (10) with $\mathfrak{n}$-valued orthonormal vector functions $\tau(t), e_1(t), \dots, e_{k-1}(t)$ and a $\mathfrak{k}$-valued vector function $P(t)$ satisfying both (11) with $c = 0$ and the conditions $\dot{e}_i + [P, e_i] = \sum_{j=1}^{k-1} \alpha_{ij} e_j + \mu_i \tau$ $(1 \leq i \leq k-1)$ for some smooth functions $\{\alpha_{ij}(t)\}$ and $\{\mu_i(t)\}$. By (2) one can replace (11) with the conditions

$$(12) \qquad [e_i, e_j] = [e_i, \tau] \equiv 0 \quad \text{for} \quad i, j = 1, \dots, k-1.$$

Again, as in the proof of Proposition 11, we notice that the $(k-1) \times (k-1)$ matrix (α_{ij}) is skew-symmetric and one can make it zero rotating the orthonormal basis in $\mathrm{Span}\,\{e_1, \dots, e_{k-1}\}$. In the same notations one obtains $\dot{e}_i + [P, e_i] = \mu_i \tau$ and (12) still holds.

We can simplify the conditions $\dot{e}_i + [P, e_i] = \mu_i \tau$ further choosing an appropriate lift $\Gamma \subset U$ of the curve $\gamma \subset N$. Namely, for the curve $\Gamma' = \Gamma V^{-1}$ where $V = V(t)$ is a smooth curve in the isotropy subgroup K and such that $dL_V^{-1}\dot{V} = P \subset \mathfrak{k}$ one has

$$e'_i = \mathrm{Ad}_V\, e_i, \quad \tau' = \mathrm{Ad}_V\, \tau, \quad P' = 0.$$

Omitting dashes one arrives at the submanifold defined by (10) with the curve $\Gamma(t)$ such that $dL_\Gamma^{-1}\dot{\Gamma} = \tau \in \mathfrak{n}$ and the vector functions $\tau, e_1, \dots, e_{k-1}$ being orthonormal and satisfying both (12) and $\dot{e}_i = \mu_i \tau$.

Now suppose that locally for $i \geq l$ $\mu_i \equiv 0$ and $\{\mu_i\}$ with $i < l$ have no zeros $(1 \leq l \leq k)$. Then the vectors $\{e_l, \dots, e_{k-1}\}$ are constant (if any) and span an abelian subalgebra $\mathfrak{u}_0 \subset \mathfrak{n} \subset \mathfrak{u}$ of dimension $k-l$. Let W be a minimal subspace in $\mathfrak{n}$ locally containing all the curves $\tau(t), e_1(t), \dots, e_{l-1}(t)$ and denote $\mathfrak{u}'$ the minimal Lie subalgebra in $\mathfrak{u}$ spanned by W and $\mathfrak{n}' = \mathfrak{n} \cap \mathfrak{u}'$ the minimal Lie triple system in $\mathfrak{n}$ containing W. By induction one can easily see that $[\mathfrak{u}', \mathfrak{u}_0] = 0$ and $\mathfrak{n}' \perp \mathfrak{u}_0$. Therefore the submanifold $\operatorname{Exp}(\mathfrak{n}' \oplus \mathfrak{u}_0)$ is a totally geodesic product of a submanifold $N' = \operatorname{Exp} \mathfrak{n}'$ and a flat submanifold $\operatorname{Exp} \mathfrak{u}_0$. Since $\tau(t) \subset \mathfrak{n}' \subset \mathfrak{u}'$ one can find a curve $R(t)$ in the Lie group $U' \subset U$ of $\mathfrak{u}'$ such that $dL_R^{-1} \dot{R} = \tau$. Now the curves Γ and R satisfy the same ODE system and therefore differ by a constant element $\Gamma_0 \in U$. Moreover, the submanifold F^l defined as $R(t)\big(\operatorname{Exp} \sum_{i=1}^{l-1} u^i e_i(t)\big) = \pi\big(R(t) \exp \sum_{i=1}^{l-1} u^i e_i(t)\big)$ is contained in N'. Therefore the submanifold

$$R(t)\big(\operatorname{Exp} \sum\nolimits_{i=1}^{k-1} u^i e_i(t)\big) = \pi\big(R(t) \exp \sum\nolimits_{i=1}^{k-1} u^i e_i(t)\big)$$
$$= \pi\big(R(t)\ \exp \sum\nolimits_{i=1}^{l-1} u^i e_i(t)\ \exp \sum\nolimits_{j=1}^{k-l} u^j e_j\big)$$

is contained in a cylinder over N' with the flat generatrix $\operatorname{Exp} \mathfrak{u}_0$. This submanifold differs from F^k only by the isometry Γ_0 of N.

If $l = 1$ we just get a cylinder over a smooth curve. Let now $l > 1$. Starting with the end we can successively choose a new orthonormal basis $f_1(t), \dots, f_{l-1}(t)$ in the space $\operatorname{Span}\{e_1(t), \dots, e_{l-1}(t)\}$ such that $\dot{f}_1 = k_1 f_2, \dot{f}_2 = -k_1 f_1 + k_2 f_3, \dots, \dot{f}_{l-2} = -k_{l-3} f_{l-3} + k_{l-2} f_{l-1}, \dot{f}_{l-1} = -k_{l-2} f_{l-2} + k_{l-1} \tau$. In orther words, $f_1, f_2, \dots, f_{l-1}, \tau$ are the first l vectors of the Frenet frame of a curve $F(t) = \int f_1(t) dt \in \mathfrak{n}'$ with non-vanishing curvatures $k_1, \dots, k_{l-1}$. The equations (12) still hold for the vectors $\{f_i\}$, that is $[f_i, f_j] = [f_i, \tau] \equiv 0$. For our purposes it is only sufficient to consider the vector function f_1. We are going to show that under some generic conditions f_1 takes values in an abelian subalgebra in $\mathfrak{n}'$. Then the same will be true for $f_2, \dots, f_{l-1}, \tau$.

More precisely, the following Lemma holds:

Lemma 3. *Let $f(t)$ be a smooth curve in a tangent space $\mathfrak{m} = T_O M$ of a symmetric space M such that $[f, \dot{f}] \equiv 0$. Then for any interval on the curve there exists a smaller interval contained in a flat Lie triple system in $\mathfrak{m}$.*

Proof of Lemma 3. If M is reducible, then both the condition and the statement of the Lemma hold for each factor, so it is sufficient to restrict oneself to an irreducible case. For a Euclidean factor there is nothing to prove, and for factors of noncompact type one can pass to their compact duals. So the Lemma is needed to be proved only for a curve in the tangent space to a symmetric space with a simple compact Lie algebra $\mathfrak{g}$. In this case the Killing-Cartan form B is strictly negatively definite on $\mathfrak{g}$ and the operators ad_g, $(g \in \mathfrak{g})$ are skew symmetric with respect to B. For a point t on the curve let $\mathfrak{g} = \mathfrak{g}_0(t) \oplus \mathfrak{g}_1(t) \oplus \dots \oplus \mathfrak{g}_m(t)$ be an orthogonal decomposition of $\mathfrak{g}$ with respect to B on eigensubspaces of the symmetric operator $\operatorname{ad}^2_{f(t)}$ with $\mathfrak{g}_0(t)$ corresponding to the eigenvalue zero. Assume that in a neighbourhood I of a point t_0 the dimensions $m_\alpha = \dim \mathfrak{g}_\alpha(t)$ are constant (for any neighbourhood one can pick a smaller one with this property). Then $\operatorname{ad}_{f(t)}$ can be decomposed as $A(t)\Phi(t)^t A(t)$, where $A(t) \in SO(\mathfrak{g})$, $A(t_0) = \mathrm{id}$, ^{t} is for transposing and $\Phi(t)_{|\mathfrak{g}_\alpha(t_0)} = \phi_\alpha(t) J_\alpha$, where

$J_\alpha : \mathfrak{g}_\alpha(t_0) \to \mathfrak{g}_\alpha(t_0)$ is a fixed operator with $J_\alpha^2 = -\mathrm{id}$, $\phi_0 \equiv 0$ and $\phi_\alpha(\alpha > 0)$ are pointwise nonequal smooth positive functions on I.

Since $[f, \dot f] = 0$ the operators ad_f and $\mathrm{ad}_{\dot f}$ commute. One has $\mathrm{ad}_{\dot f} = (\mathrm{ad}_f)^{\cdot} = A(\dot\Phi + [K, \Phi])^t A$, where $K = {}^t A \dot A$ is a skew symmetric operator on $\mathfrak{g}$ and the brackets are for the Lie brackets in $\mathfrak{gl}(\mathfrak{g})$. Since $\dot\Phi$ commutes with Φ one gets $[[K, \Phi], \Phi] = 0$ which implies $[K, \Phi] = 0$ on I. It follows that the subspaces $\mathfrak{g}_\alpha(t_0)$ are invariant subspaces of K and the restrictions of K to $\mathfrak{g}_\alpha(t_0)$ with $\alpha > 0$ commute with J_α. Then $\frac{d}{dt}[\mathrm{ad}_{f(t_0)}, \mathrm{ad}_{f(t)}] \equiv 0$, so $[\mathrm{ad}_{f(t_0)}, \mathrm{ad}_{f(t)}] \equiv 0$ on I. Therefore $[f(t_0), f(t)] \equiv 0$. Since $t_0 \in I$ can be chosen arbitrary, the vectors $\{f(t)\}_{t \in I}$ pairwise commute. Then their linear span is an abelian subalgebra in $\mathfrak{g}$. □

Returning to the proof of Proposition 12, we get a submanifold F^l in the symmetric space N' defined as $R(t)(\mathrm{Exp}(\sum_{i=1}^{l-1} u^i f_i(t))) = \pi(R(t)\exp(\sum_{i=1}^{l-1} u^i f_i(t)))$, and the vector functions $f_1, \dots, f_{l-1}, \tau = dL_R^{-1}\dot R$ take values in a fixed abelian subgalgebra $\mathfrak{a} \subset \mathfrak{u}'$. Then up to the left action of a fixed element of U' the curve $R(t)$ is contained in the abelian subgroup $A \subset U'$ of the subalgebra $\mathfrak{a}$. The same is true for the submanifold $R(t)\exp(\sum_i u^i f_i)$ and therefore its projection F^l lies in a flat totally geodesic submanifold of N'. □

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