MAT4TOP: Admin Overview

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- student classroom presentations worth 20% of assessment.
- 2 hour exam worth 50% of assessment.

Outline

Indexed sets

We will often need to discuss sets of subsets of a given set A. If the set of sets contains a small number of sets we can write down all members in a list, say U_1, U_2, U_3, U_4. Where we have a countably infinite set of sets, we could write $W_1, W_2, \ldots, W_n, \ldots$. However, for uncountably many sets, we cannot write them in a list. To avoid such problems we introduce notation which covers all cases.

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Definition 1 (Indexed Sets of Sets)

Let A be a set. We say that $\{U_i : i \in I\}$ is an indexed set of subsets of A iff $U_i \subseteq A$ for all $i \in I$. The set I is called the index set for the set of sets.

Intersection and Union

We can extend familiar ideas about intersection and union to arbitrary indexed set of sets.

Definition 2 (Unions and Intersections)

Let $\{U_i : i \in I\}$ be a set of subsets of a set A. We define union and intersection as follows.

$$\bigcup_{i \in I} U_i = \{x : x \in U_i \text{ for some } i \in I\} \quad \text{(Union)}$$

$$\bigcap_{i \in I} U_i = \{x : x \in U_i \text{ for all } i \in I\} \quad \text{(Intersection)}.$$

The distributive law

The following lemma generalizes a familiar distributive law of set theory.

Lemma 1

Let $\{U_i : i \in I\}$ *be a set of subsets of a set* A *and let* $B \subseteq A$. *Then*

$$\left[\bigcup_{i \in I} U_i\right] \cap B = \bigcup_{i \in I} (U_i \cap B).$$

Complements and De Morgan's laws

Definition 3 (Complements)

Suppose A and B are sets. Then the complement of B in A is given by

$$A \setminus B = \{x \in A : x \notin B\}.$$

Theorem 1 (De Morgan's Laws)

Let $\{U_i : i \in I\}$ be a set of subsets of a set A. Then

$$A \setminus \left(\bigcup_{i \in I} U_i \right) = \bigcap_{i \in I} (A \setminus U_i) \quad \textit{and} \quad A \setminus \left(\bigcap_{i \in I} U_i \right) = \bigcup_{i \in I} (A \setminus U_i).$$

This just says that the complement of a union is the intersection of the complements, and vice versa.

Cartesian products

Definition 4 (Cartesian Product)

Let A and B be sets. Then the **Cartesian product** of A and B is defined as follows.

$$A \times B = \{(x, y) : x \in A \quad \text{and} \quad y \in B\}$$

If $U \subseteq A$ and $V \subseteq B$ then $U \times V$ is just the subset

$$\{(x, y) : x \in U \text{ and } y \in V\}$$

of $A \times B$. If either U or V is empty then so is $U \times V$. The next lemma says that an intersection of Cartesian products is equal to the Cartesian product of the intersections.

Lemma 2

$(S \times X) \cap (T \times Y) = (S \cap T) \times (X \cap Y).$

The use of Cartesian products of index sets enables us to express a more general distributive law for sets than the one we gave above.

Theorem 2 (Generalized Distributive Law)

Let $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$ be families of subsets of a set A. Then

$$\left[\bigcup_{i\in I} U_i\right] \cap \left[\bigcup_{j\in J} V_j\right] = \bigcup_{(i,j)\in(I\times J)} (U_i \cap V_j).$$

Power Sets

Definition 5 (Power Set)

Let A be a set. The power set of A is the set of all subsets of A and is denoted $\mathcal{P}(A)$.

$\mathcal{P}(A)$ always includes both $\varnothing$ and A itself. If A is finite with n elements then $\mathcal{P}(A)$ has 2^n elements. For this reason the power set is sometimes denoted 2^A. The definition of $\mathcal{P}(A)$ still makes perfectly good sense when A is infinite.

Image

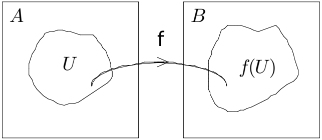
In topology the way in which sets behave under maps is of the utmost importance.

Definition 6 (Image of a Set)

Let A and B be sets, let $f : A \to B$ and let $U \subseteq A$. The image of U under f is given by

$$f(U) = \{f(x) : x \in U\} = \{y \in B : y = f(x) \text{ for some } x \in U\}.$$

The image $f(U)$ contains the images of all points in U. Note that $f(U)$ is a subset of the codomain of f.



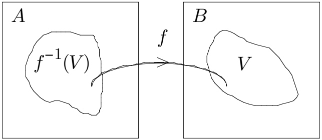
Preimage

Definition 7 (Preimage of a Set)

Let A and B be sets, let $f : A \to B$ and let $V \subseteq B$. The preimage of V under f is given by

$$f^{-1}(V) = \{x \in A : f(x) \in V\}.$$

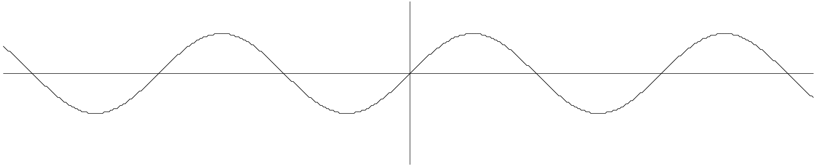
The preimage $f^{-1}(V)$ contains all points in A that map into V. Note that $f^{-1}(V)$ is a subset of the domain of f.



Note that $f^{-1}(V)$ is not the image of V under the inverse of f. For many of the maps we study, f^{-1} will not exist, even though $f^{-1}(V)$ is perfectly well defined.

Preimage Examples

Let f be the map $\sin : \mathbb{R} \to \mathbb{R}$



$$f^{-1}(\{0\}) = \{k\pi : k \in \mathbb{Z}\}$$
$$f^{-1}([0,\infty)) = \bigcup_{k\in\mathbb{Z}} [2k\pi, (2k+1)\pi]$$
$$f^{-1}([1,\infty)) = \{2k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}$$
$$f^{-1}((1,\infty)) = \varnothing$$

Lemma 3

Let A and B be sets, let $f : A \to B$, let $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$ be indexed sets of subsets of A and B respectively. Then

1. $f^{-1}\left(\bigcup_{j\in J} V_j\right) = \bigcup_{j\in J} f^{-1}(V_j)$

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4. $f\left(\bigcap_{i\in I} U_i\right) \subseteq \bigcap_{i\in I} f(U_i)$

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4. $f\left(\bigcap_{i \in I} U_i\right) \subseteq \bigcap_{i \in I} f(U_i)$
5. *If f is one-to-one then* $f\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f(U_i)$.

One-to-one and Onto

Recall from first year:

Definition 8 (One-to-one and onto Functions)

Let A and B be sets, and let $f : A \to B$.

1. $f : A \to B$ is said to be one-to-one (or injective, or an injection) iff

$$f(x) = f(y) \implies x = y,$$

for all $x, y \in A$.

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$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y.$$
3. $f : A \to B$ is said to be bijective (or a bijection) iff f is both one-to-one and onto.

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Let A and B be sets, let $f : A \to B$, *let* $U, V \subseteq A$ *and let* $X, Y \subseteq B$. *Then*

1. $U \subseteq V \Rightarrow f(U) \subseteq f(V)$

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3. $U \subseteq f^{-1}(f(U))$

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4. *If f is one-to-one then $U = f^{-1}(f(U))$*
5. $f(f^{-1}(X)) \subseteq X$

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5. $f(f^{-1}(X)) \subseteq X$
6. *If f is onto then $f(f^{-1}(X)) = X$*
7. $f^{-1}(B \setminus X) = A \setminus f^{-1}(X)$.

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8. $f^{-1}(f(A)) = A$

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7. $f^{-1}(B \setminus X) = A \setminus f^{-1}(X)$.
8. $f^{-1}(f(A)) = A$
9. *If f is one-to-one then* $f(U \setminus V) = f(U) \setminus f(V)$.

Restriction of a map

Definition 9 (Restriction of a map)

Let A and B be sets, let $f : A \to B$ and let $X \subseteq A$. The restriction $f|_X : X \to B$ of f to X is the map with domain X which agrees with f at every point of X.

The definition just says that $f|_X$ is the unique map obtained by just restricting the domain of f to X.

Partitions

A topology is a set of subsets, subject to certain rules. Another important set of subsets in mathematics is a partition. Topologies are not partitions, but we will encounter partitions in this course, so it's good to clarify exactly what they are.

Definition 10 (Partition)

Let S be a set. A set $\mathcal{P}$ of non-empty subsets of S is called a partition of S provided

(i) $(\forall x \in S)(\exists B \in \mathcal{P})\ x \in B$

(ii) $(\forall B, C \in \mathcal{P})\ B \neq C \Rightarrow B \cap C = \varnothing$

Equivalence Relations

A notion very closely related to partitions is that of equivalence relations. Recall that a relation $\sim$ on a set A is an equivalence relation if the following three conditions are satisfied:

1. $x \sim x$, for all $x \in A$ (reflexive).
2. $x \sim y \iff y \sim x$, for all $x, y \in A$ (symmetric).
3. $x \sim y, y \sim z \implies x \sim z$, for all $x, y, z \in A$ (transitive).

Given an equivalence relation and $x \in A$, the equivalence class $[x]$ is the set $[x] := \{y \in A : y \sim x\}$. Clearly, the set of equivalence classes forms a partition of A. Conversely, given a partition $\mathcal{P}$ of A, there is a corresponding equivalence relation: we set $x \sim y$ if x, y belong to the same part of $\mathcal{P}$.