Outline

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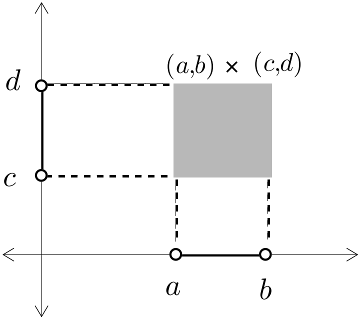
Some open sets in $\mathbb{R}^2$ – the open rectangles – are Cartesian products of open sets in $\mathbb{R}$ – the open intervals.



Defining the Product Topology (of two spaces)

Lemma 1

Let $(A, \mathcal{S})$ *and* $(B, \mathcal{T})$ *be topological spaces. Then* $\mathcal{B} = \{U \times V : U \in \mathcal{S}, V \in \mathcal{T}\}$ *is a base for a topology on* $A \times B$*.*

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Definition 1 (Product Topology)

The topology generated by the base given in Lemma 1 is called the product topology on $A \times B$.

Coordinate Systems

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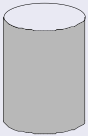
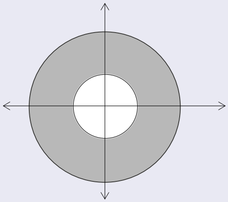
$[0, 1] \times S^1$ is homeomorphic to a cylinder in $\mathbb{R}^3$ or an annulus in $\mathbb{R}^2$.





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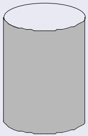

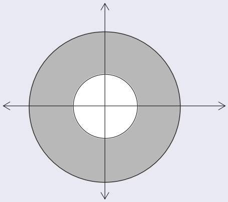

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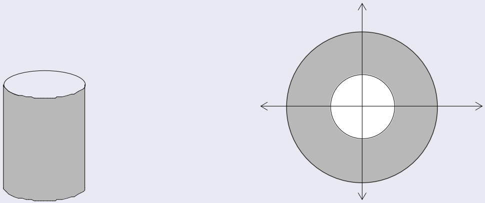


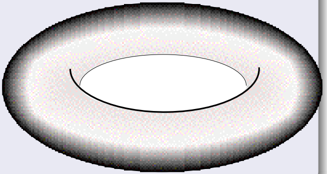
For $(x, y) \in [0,1] \times S^1$:

- x is the vertical coordinate on the cylinder or the radial coordinate in the annulus.
- y is the “circular" coordinate in both cases.

Another Example

Example 2

$S^1 \times S^1$ is homeomorphic to a torus in $\mathbb{R}^3$. For $(x, y) \in S^1 \times S^1$, the entries x and y are the "coordinates" in each of the "circular" directions.



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Theorem 2 (Continuity via Components)

Let $(X, \mathcal{R}), (A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces and suppose $f : X \to A \times B$. Then $f = (f_A, f_B)$ is continuous $\iff$ $f_A : X \to A$ and $f_B : X \to B$ are both continuous.

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Definition 3 (Fibres)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces. For $x \in A$ and $y \in B$ the subspaces $\{x\} \times B$ and $A \times \{y\}$ of $A \times B$ are called fibres.

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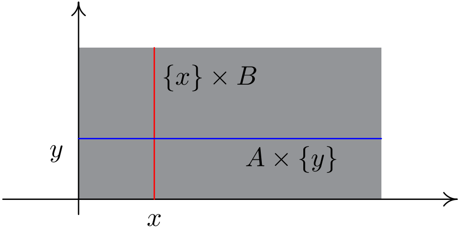

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Exercise: prove that path-connectedness, Hausdorffity, separability, metrisability, and the first and the second countability are preserved under taking products (of two sets).

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Suppose $\mathcal{C}$ is an open cover for $A \times B$ with no finite subcover. Then there exists $y_0 \in B$, such that no cylinder $A \times V$, where $V \subset B$ is open and $y_0 \in V$, has a finite subcover. If not, then for all $y \in B$ there would exists an open $V_y \subset B$ such that $A \times V_y$ has a finite subcover. $\{V_y\}$ is an open cover for B, so it contains a finite subcover, and so on. Next, there exists $x_0 \in A$, such that no open box $U \times V \ni (x_0, y_0)$, has a finite subcover. If not, then for all $x \in A$ there would exists an open box $U_x \times V_x \ni (x, y_0)$ having a finite subcover. $\{U_x\}$ is an open cover for A, so it contains a finite subcover $\{U_{x_i}\}_{i=1}^n$. Then $V = \cap_{i=1}^n V_{x_i}$ is open in B and contains y_0. And so $\cup_{i=1}^n (U_{x_i} \times V_{x_i})$ is an open cover for the cylinder $A \times V$, contradicting the choice of y_0.
But (x_0, y_0) is covered by something, which contains a box (as boxes form a base).

Compactness

Theorem 4 (Tychonoff)

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But (x_0, y_0) is covered by something, which contains a box (as boxes form a base). Contradiction. □

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- Theorems 3 and 4 remain valid for infinite products.

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Let $C_0 = [0, 1]$. To obtain C_1, we remove from C_0 the middle open third, so $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. To obtain C_2, we remove from C_1 the middle open third of each interval, so $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. And so on .... The set $C = \cap_{n=0}^{\infty} C_n$ is called the Cantor set.

- The Cantor set is small:
  1. it is compact (Heine-Borel).
  2. it has Lebesgue measure zero (but we can modify the construction to get a Cantor set of positive measure; even arbitrarily close to 1).
  3. it is nowhere dense (= its complement is dense).
- The Cantor set is large:
  1. it is nonempty (e.g. C contains 0, or $\frac{1}{3}$, or any endpoint of any interval we're removing). Anything more interesting?
  2. it can be constructively described as follows. Represent every number from $[0, 1]$ to base 3. Then C is the set of those numbers which can be written using only digits 0 and 2. This includes e.g. $0.222\ldots_3 = 1,\ 0.0222\ldots_3 = 0.1_3 = (\frac{1}{3})_{10}$, etc. Example: $0.020202\ldots_3 = (\frac{2}{9} + \frac{2}{81} + \frac{2}{729} + \ldots)_{10} = (\frac{1}{4})_{10} \in C$.
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