Outline

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Theorem 1

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Uses the continuity of the "support line".
A support line is somewhat similar to the tangent line.

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Theorem 2 (Borsuk)

The Conjecture is true on the plane: any compact of diameter 1 *can be cut into three pieces of a smaller diameter.*

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Inscribe in a regular hexagon of "width 1". First in a rhombus with angle 60^0, then Intermediate Value Theorem.
Then cut into three pieces. □

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“Obvious", but quite nontrivial to prove in full: the curve can be very “wiggly".
Not true on a torus (or on something more complicated).

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Proof for polygonal curves.

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And moreover, all the points which are "far enough" belong to A_0. This is the exterior. The other domain is the interior. □

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The map f is NOT injective. If it were, then $[0,1]$ and $[0,1] \times [0,1]$ would be homeomorphic. They are not: the square has no cut points.