Outline

1. Classification of surfaces

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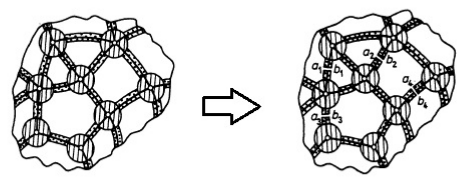

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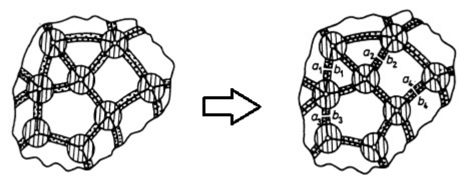

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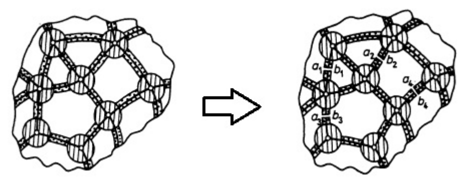

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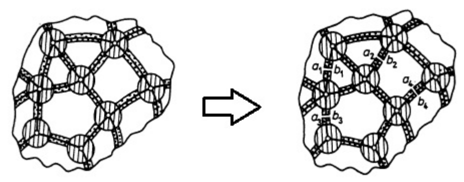

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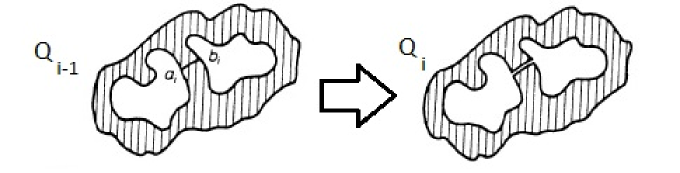

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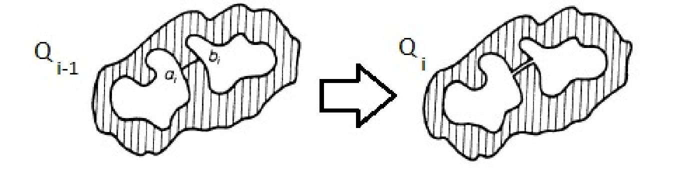

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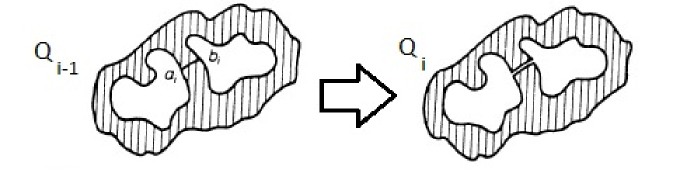

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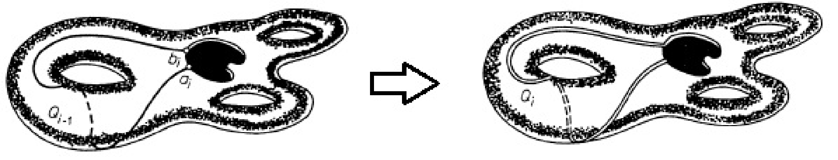

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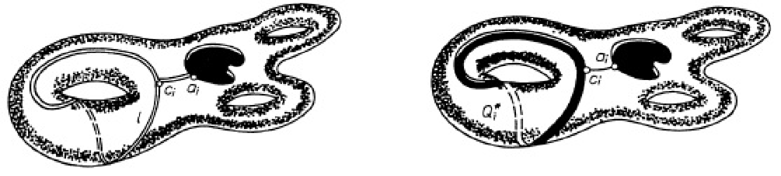

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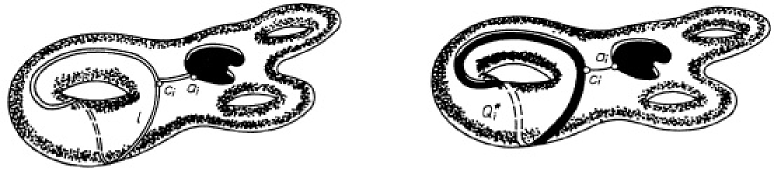

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This proves the induction step.

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