Outline

1. Fundamental group: definition
2. Fundamental group: computation
3. Fundamental group: applications

Homotopy

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Definition 1 (Path; Definition 4 of Lecture 5)

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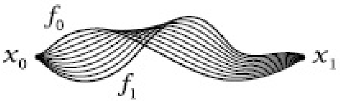

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$F(s,t) = f((1-t)s + t\phi(s))$. □

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The equivalence class $[f]$ of a paths f is called the homotopy class of f.

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Suppose $f, g : [0,1] \to A$ are paths such that $f(1) = g(0)$. Then the product of the paths f and g is the path $h = f \circ g$ defined by

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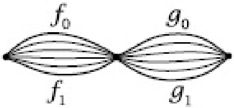

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2. To prove the theorem we need to verify the axioms of the group: associativity, identity and inverse.

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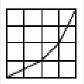

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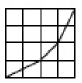

4. Define the identity e to be the class of the constant map $c : [0,1] \to A,\ c(s) = x_0$. Then for a loop f we have $c \circ f \cong f \circ c \cong f$ by reparametrisations below and hence by Lemma 1.

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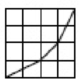

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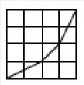

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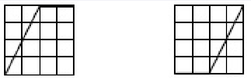

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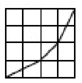

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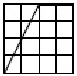

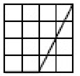

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Let h be a path joining x_0 and x_1. Define the map $\beta_h : \pi_1(A, x_0) \to \pi_1(A, x_1)$ by $\beta_h[f] = [h \circ f \circ \bar{h}]$.

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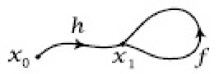

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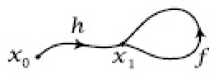

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So for a path connected space we can simply speak of $\pi_1(A)$.

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