Outline

1. Metric Spaces
2. Open and Closed Sets
3. Subspaces

Metrics

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(M3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

We say that (A, d) is a metric space.

Metrics from norms

In $\mathbb{R}$, or more generally $\mathbb{R}^n$, the elementary definitions of continuity and limits use the Euclidean norm:

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- We can have two different metrics on the same set. (eg, Euclidean & discrete metrics on $\mathbb{R}^2$.)
- Although we study non-Euclidean metrics, your intuition about distance should still work.
- When proving a general metric space result, you can only use properties (M1),(M2) and (M3).

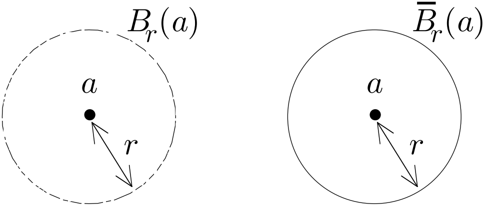
Balls

Definition 4 (Open and Closed Balls)

Let (A, d) be a metric space, $a \in A$, $r > 0$. We define the open and closed balls with centre a, radius r, as follows.

$B_r(a) = \{x \in A : \ d(x, a) < r\}$ (Open ball)

$\overline{B}_r(a) = \{x \in A : d(x, a) \leq r\}$ (Closed ball)



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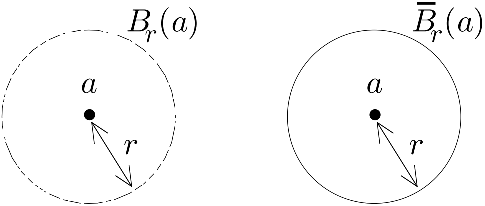

These definitions make sense in any metric space.

It's all balls

The following elementary lemma is often useful.

Lemma 1 (Union of Open Balls)

Every metric space (A, d) *is a union of open balls. That is, for each* $x \in A$ *we can choose* $r_x > 0$ *so that* $A = \bigcup_{x \in A} B_{r_x}(x)$.

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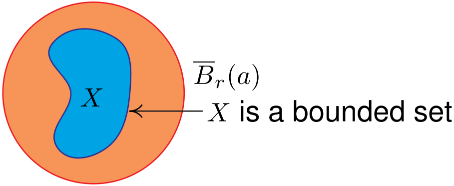
Proof.

For each $x \in A$, choose $r_x > 0$ randomly (let $r_x = 1$ if you like). By definition $B_{r_x}(x) \subseteq A$ for each $x \in A$, so $\bigcup_{x \in A} B_{r_x}(x) \subseteq A$. On the other hand, for each $x \in A$, we have $x \in B_{r_x}(x)$, so $A \subseteq \bigcup_{x \in A} B_{r_x}(x)$. Thus $A = \bigcup_{x \in A} B_{r_x}(x)$. □

Bounded sets

Definition 5 (Bounded Set)

$X \subseteq A$ is bounded iff $X \subseteq \overline{B}_r(a)$ for some $r > 0$, $a \in A$.



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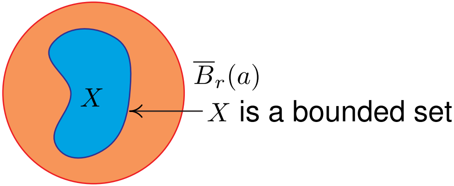
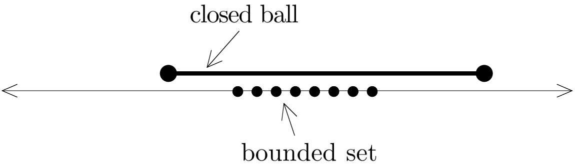
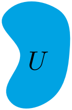


This generalizes the idea of a bounded subset of $\mathbb{R}$:



Open sets

Open sets contain none of their boundary points.



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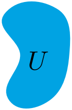

Definition 6 (Open set)

Let (A, d) be a metric space. $U \subseteq A$ is an open set iff $(\forall a \in U)(\exists r > 0)\ B_r(a) \subseteq U$.

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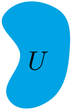

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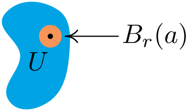

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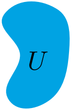

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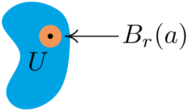
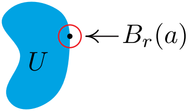


A ball around a boundary point of an open U can't stay inside U.



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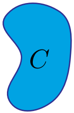

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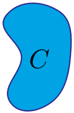

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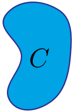

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- If V is closed $A \setminus V$ has no boundary points as they must all be in V. Hence $A \setminus V$ is open.

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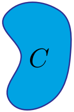

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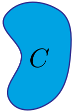

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- The notation V' for the complement is also acceptable but $\overline{V}$ is not!

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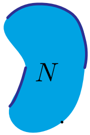

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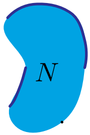

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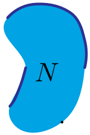

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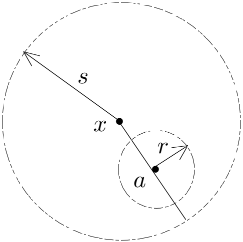

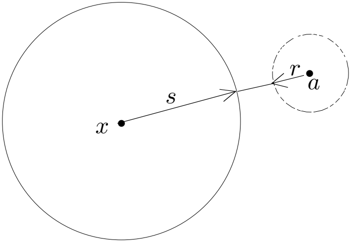

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- The importance of these properties cannot be overstated, as we will see in the next lecture.

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- Strictly speaking, this new metric on X is the restriction of $d : A \times A \to \mathbb{R}$ to $X \times X$.
- Many useful subsets of $\mathbb{R}^n$ (eg spheres, circles, curves, balls, tori, fractals) are now metric spaces at no extra cost.

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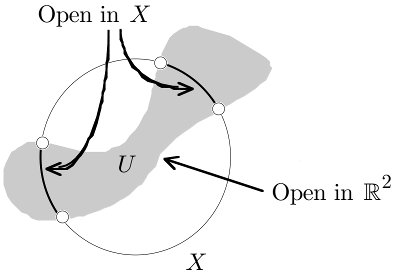

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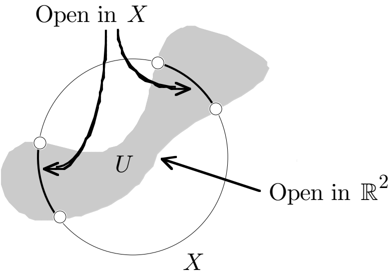

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Lemma 2 (Open balls in Metric Subspaces)

Let (X, d) be a subspace of (A, d). Let $x \in X$ and $r > 0$. Then $B_r^X(x) = X \cap B_r^A(x)$.

A useful lemma

First we need some notation. Let (X, d) be a subspace of (A, d). For $x \in X$ and $r > 0$, we will denote the open ball of radius r about x in (X, d) by $B_r^X(x)$ and we will denote the open ball of radius r about x in (A, d) by $B_r^A(x)$.

Lemma 2 (Open balls in Metric Subspaces)

Let (X, d) be a subspace of (A, d). Let $x \in X$ and $r > 0$. Then $B_r^X(x) = X \cap B_r^A(x)$.

Proof.

$$\begin{aligned} B_r^X(x) &= \{y \in X : d(x, y) < r\} \\ &= \{y \in A : y \in X\} \cap \{y \in A : d(x, y) < r\} \\ &= X \cap B_r^A(x). \end{aligned}$$

□

Proof of Theorem 4

First suppose that V is open in X. Then for every $x \in V$ there is an $r_x > 0$ such that $B^X_{r_x}(x) \subseteq V$. The set $U = \bigcup_{x \in V} B^A_{r_x}(x)$ is open in A, by Theorem 3 (OS2). Furthermore,

$$\begin{aligned} V &= \bigcup_{x \in V} B^X_{r_x}(x) && \text{(by Lemma 1)} \\ &= \bigcup_{x \in V} X \cap B^A_r(x) && \text{(by Lemma 2)} \\ &= X \cap \bigcup_{x \in V} B^A_r(x) && \text{(by Lemma 1 of Lecture 1)} \\ &= X \cap U && \text{(by definition of } U\text{).} \end{aligned}$$

Conversely, suppose that U be open in A, let $V = X \cap U$ and let $x \in V$. Since U is open in A, there is an $r > 0$ such that $B^A_r(x) \subseteq U$. Thus $X \cap B^A_r(x) \subseteq X \cap U$. Hence by Lemma 2, $B^X_r(x) \subseteq V$. Thus V is open in X. □