Outline

1. Continuity in metric spaces
2. Topological Spaces
3. Metrisability

For metric spaces (A, d), (B, p) and $f : A \to B$ the definition of "f is continuous at $a \in A$" can be written in several ways:

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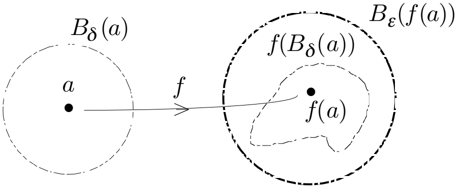

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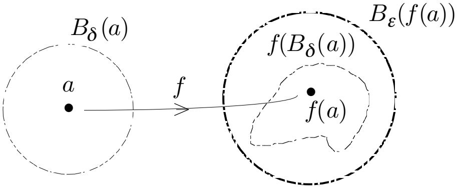

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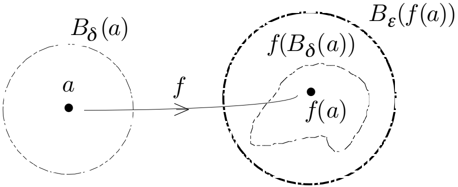

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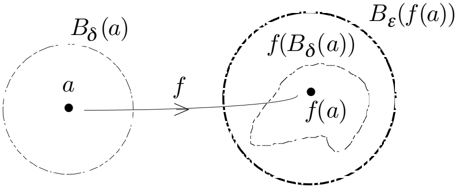

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Notice that (C1), (C2) and (C3) all make sense in metric spaces.

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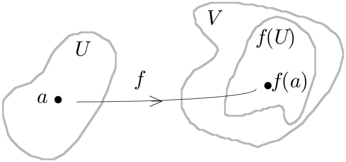

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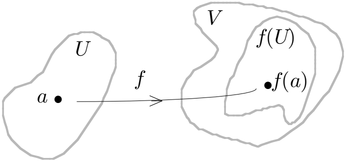

Lemma 1 (Equivalence of definitions)

All four definitions (C1), (C2), (C3) and (C4) are equivalent.

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- Sets of sets with these properties can often be defined without a metric.
- The basic idea of topology is to take these properties as axioms.

Topology

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- By Theorem 3 of Lecture 2, everything we prove about topological spaces holds for metric spaces.

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(CS3) *If* C *and* D *are closed,* $C \cup D$ *is closed.*

(CS3′) *If* $\{C_i : i \leq n\}$ *is a finite set of closed sets* $\bigcup_{i \in I} C_i$ *is closed.*

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- Differs from the Euclidean topology on $\mathbb{R}^2$ (different open sets).

Seeing that a set is open

Regardless of what topology we are using, we are working with open sets. The following result is often useful in determining whether a given set is open.

Lemma 2

Let $(A, \mathcal{S})$ be a topological space. Then a subset $B \subseteq A$ is open iff for every $x \in B$, there is an open set $U_x \in \mathcal{S}$ with $x \in U_x \subseteq B$.

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- Are there not-metrisable topologies at all?

Half-interval (Sorgenfrey) Topology

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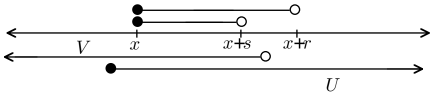

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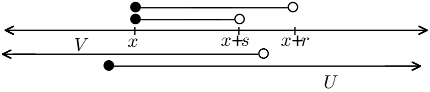

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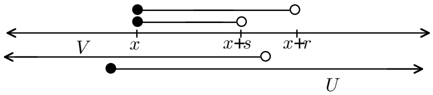

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- The half interval topology is not metrisable.

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- This topology is non-metrisable if A is infinite.