Outline

Continuity in topological spaces

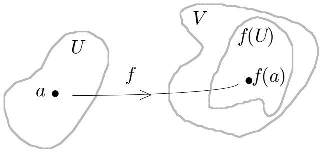
Lemma 1 of Lecture 3 gives a definition that works in any space:

Definition 1 (Continuity)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces. $f : A \to B$ is continuous at $a \in A$ provided it satisfies

- If V is open and $f(a) \in V$, then there is an open U such that $a \in U$ and $f(U) \subseteq V$.

f is continuous if it is continuous at each point in A.



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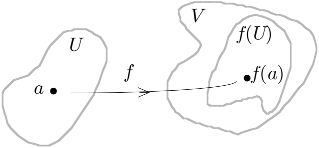

- Definition of continuity involves two topologies.

Continuity and Preimages

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Theorem 1 gives us a succinct equivalent way of rephrasing continuity. Indeed, this is the modern definition of continuity. To stress this important notion, we rewrite the definition:

Definition 2 (The Modern Definition of Continuity)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces. Then $f : A \to B$ is said to be continuous if $f^{-1}(V)$ is open in $(A, \mathcal{S})$ for each open set V in $(B, \mathcal{T})$.

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We can express the idea in this definition in simple terms by saying that a map is continuous iff the preimages of open sets are open. Note that this characterizes continuity in terms of open sets without reference to balls, metrics or messy inequalities.

Reaping the rewards

One of the big advantages of the modern definition of continuity is that now many results are completely trivial. This is the case of the following:

Theorem 2 (Continuity of Composites)

Let $(A, \mathcal{S}), (B, \mathcal{T})$ and $(C, \mathcal{U})$ be topological spaces. If $f : A \to B$ and $g : B \to C$ are continuous then $g \circ f : A \to C$ is continuous.

$\odot$

Initially ...

Suppose that $(A, \mathcal{S})$ is a topological space, let B be a set, and let $f : B \to A$ be a function. The idea is to define a natural topology on B that makes f continuous. This is done by deciding that a subset U of B is open iff $U = f^{-1}(V)$ for some open set $V \in \mathcal{S}$. This is called the initial topology on B and we denote it $\mathcal{I}_f$. Its construction is chosen so that f is continuous, as a function from $(B, \mathcal{I}_f)$ to $(A, \mathcal{S})$.

Definition 3 (Initial Topology)

Suppose that $(A, \mathcal{S})$ is a topological space, let B be a set, and let $f : B \to A$ be a function. Then

$$\mathcal{I}_f := \{U \subseteq B : U = f^{-1}(V), \text{ for some } V \in \mathcal{S}\}$$

is a topology on B called the initial topology.

... and Finally

The final topology is defined in a similar way, by reversing the arrows.

Definition 4 (Final Topology)

Suppose that $(A, \mathcal{S})$ is a topological space, let B be a set, and let $f : A \to B$ be a function. Then

$$\mathcal{T}_f := \{U \subseteq B : f^{-1}(U) \in \mathcal{S}\}$$

is a topology on B called the final topology.

Of course, we have to prove that the initial and final topologies really are topologies, and the function f is continuous in each case; this is left to the exercises/assignments.

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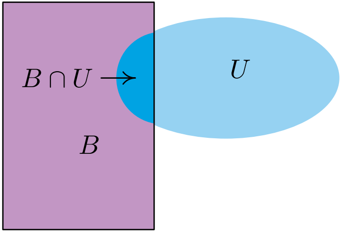

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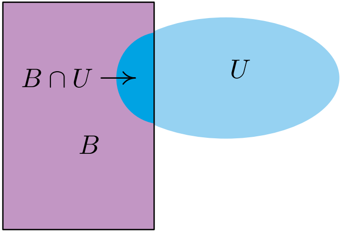

Definition 5 (Subspace Topology = Induced Topology = Relative Topology)

Let $(A, \mathcal{S})$ be a topological space and let $B \subseteq A$. Then $\mathcal{T} = \{U \cap B : U \in \mathcal{S}\}$ is a topology on B called the subspace topology. We say that $(B, \mathcal{T})$ is a subspace of $(A, \mathcal{S})$.

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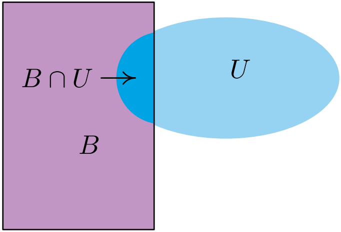

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Of course, we must verify that $\mathcal{T}$ is a topology on B.

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Example 1

Let $X = [0,1] \cup [3,4]$. Although $[0,1]$ and $[3,4]$ aren't open in $\mathbb{R}$, they are open in X because $(-1,2)$ and $(2,5)$ are open in $\mathbb{R}$ and

$$[0,1] = X \cap (-1,2) \qquad \text{and} \qquad [3,4] = X \cap (2,5)$$

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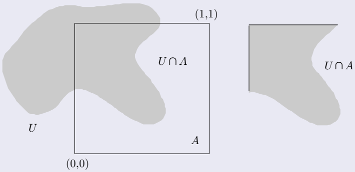
Example 2

Subspace $A = [0,1] \times [0,1]$ of $\mathbb{R}^2$ and U a “typical" open subset of $\mathbb{R}^2$.

- $U \cap A$ open in A, but seems to have “boundary" points.
- They are only boundary points of $U \cap A$ considered as a subset of $\mathbb{R}^2$.



Closed in a subspace

Theorem 3 (Closed Sets in Subspaces)

Let $(A, \mathcal{S})$ *be a topological space and* $B \subseteq A$*.* C *is closed in* B *iff* $C = D \cap B$ *for some closed* $D \subseteq A$*.*

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Warning

Intuition about open and closed sets not always correct in subspaces.

Restrictions and Continuity

Definition 6 (Restriction of a map, Definition 1 of Lecture 1)

Let A and B be sets, $f : A \to B$ and $X \subseteq A$. The restriction $f|_X : X \to B$ of f to X is the map with domain X which agrees with f at every point of X.

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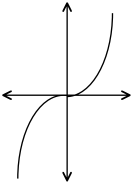
Theorem 4 (Continuity of Restrictions)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces and let $X \subseteq A$. If $f : A \to B$ is continuous then $f|_X$ is continuous.

When using glue . . .

We often define a function by a “branched" formula. EG:

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} -x^2 & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$



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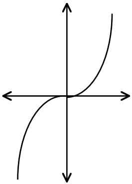

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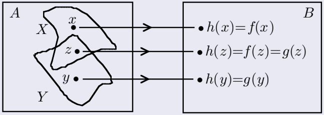
Theorem 5 (Glueing Continuous Functions)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces, let X and Y be closed in A and let $f : X \to B$ and $g : Y \to B$ be continuous. Provided f and g agree on $X \cap Y$ the following map is well defined and continuous.

$$h : X \cup Y \to B : x \mapsto \begin{cases} f(x) & \textit{if } x \in X \\ g(x) & \textit{if } x \in Y \end{cases}$$



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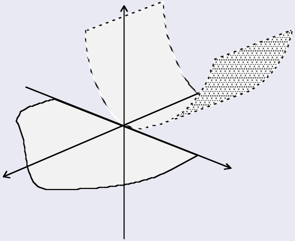
Example 4

$$X = \{(x, y) : x \geq 0\},$$
$$Y = \{(x, y) : x < 0\} \cup \{(0, 0)\}$$
$$f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \begin{cases} 0 & \text{if } (x, y) \in X \\ y^2 & \text{if } (x, y) \in Y \end{cases}$$



is continuous on X and Y but not on the whole $\mathbb{R}^2$.

We have already seen the discrete topology; this is the topology for which every subset is open. At the other end of the spectrum, there is the indiscrete (or trivial, or coarse) topology, which has the fewest possible open sets.

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The indiscrete topology on a set A is the topology for which $\varnothing$ and A are the only open sets.

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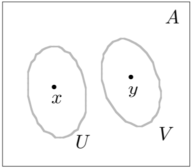
One of the most useful properties that ensures that spaces are mathematically interesting is the Hausdorff property:

Definition 8 (Hausdorff Spaces)

We say a space $(A, \mathcal{S})$ is Hausdorff if for any two distinct points $x, y \in A$ there exist disjoint open U and V such that $x \in U$ and $y \in V$.



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Many interesting spaces are Hausdorff. In particular, we have:

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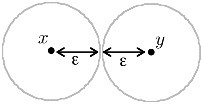
Every metric space (A, d) is Hausdorff.



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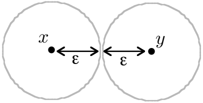

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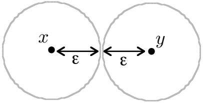

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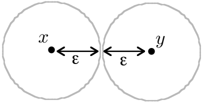

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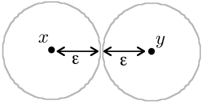

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- It is easy to see that the finite-closed topology on A is not Hausdorff if A is infinite (and so in this case, the topology is not metrisable).
- The indiscrete topology on A is not Hausdorff if A has two or more elements (and so in this case, the topology is not metrisable).
- It is easy to see that the half-interval topology is Hausdorff. (It is not metrisable, but this is not so easy to see.)

The notion of topology developed from ideas of Cantor, Poincaré, Frèchet and others, a little over 100 years ago. Our modern definition developed from work by the German mathematician Felix Hausdorff and the Hungarian mathematician Frigyes Riesz. Hausdorff was a great intellectual. Initially interested in becoming a music composer, he made important contributions to mathematics, astronomy and philosophy, and wrote literary works, as well as a book of poems. Both Hausdorff and Riesz were Jewish; when Hausdorff learnt that he was to be sent to a concentration camp, he, his wife and his wife's sister committed suicide together, on January 25, 1942.

Felix Hausdorff