Outline

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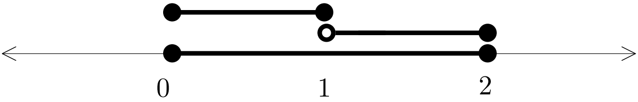

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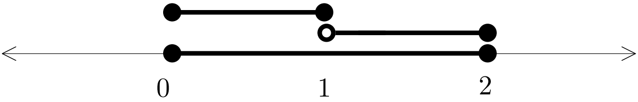

- We need to define connectedness topologically.

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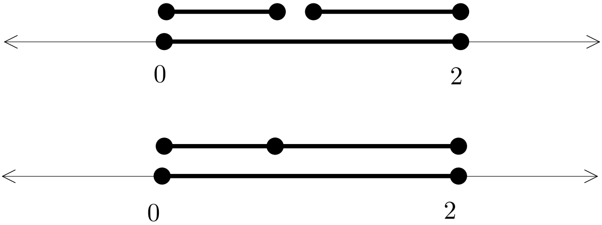

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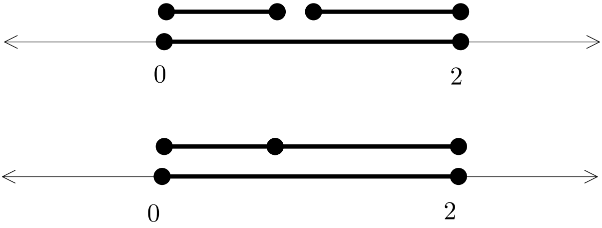

- In fact, we can't write $[0, 2]$ as the union of two disjoint non-empty closed sets of any kind.

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- We can't write Y as the union of two disjoint closed subsets of $\mathbb{R}$.

Subspaces to the rescue!

If we view $(0, 1)$ and $(3, 4)$ as subsets of the subspace Y then $(0, 1)$ and $(3, 4)$ are closed in Y since

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Definition 1 (Connected Sets)

A topological space $(A, \mathcal{S})$ is disconnected if it is a union of two disjoint non-empty closed subsets and connected otherwise. ($X \subseteq A$ is disconnected if it is disconnected as a subspace and connected otherwise.)

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Definition 2 (Clopen Partitions)

A set which is both open and closed is called clopen. If a topological space $(A, \mathcal{S})$ is the union of disjoint non-empty clopen sets C and D, we call $\{C, D\}$ a clopen partition of $(A, \mathcal{S})$.

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A space A is disconnected if there exists a clopen partition of A.

Connectedness in $\mathbb{R}$

- Intervals come in several varieties:
  $(a, b),\ (a, \infty), (-\infty, b), (-\infty, \infty) = \mathbb{R}$,
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Definition 3 (Intervals)

A subset J of $\mathbb{R}$ is an interval iff for every $s, t \in J$ with $s < t$, every $x \in \mathbb{R}$ satisfying $s \leq x \leq t$ is also in J.

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Theorem 1 does not hold for non Euclidean topologies on $\mathbb{R}$, eg, half-interval or discrete topology.

Theorem 2 (Vector Spaces Connected)

Every normed vector space is connected.

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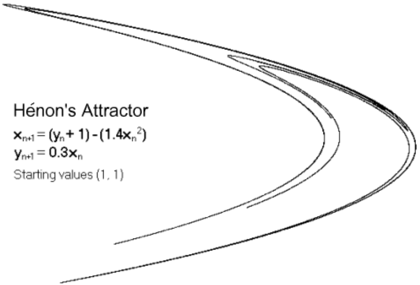
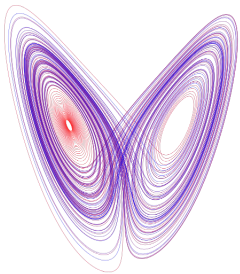
- No simple characterisation of connected subsets of $\mathbb{R}^n$ for $n > 1$.
- Even in $\mathbb{R}^2$ and $\mathbb{R}^3$, connected sets can be very complicated.





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    - Since both possibilities lead to a contradiction, $\mathbb{R}$ must be connected.

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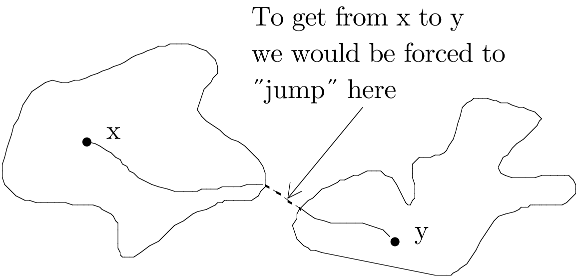

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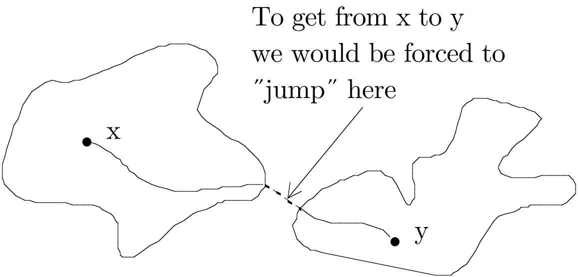

- This suggests a definition using continuous maps because they can’t “jump".

Paths

Definition 4 (Path)

Let $(A, \mathcal{S})$ be a topological space and let $x, y \in A$. A path from x to y in $(A, \mathcal{S})$ is a continuous map $p : [0, 1] \to A$ such that $p(0) = x$ and $p(1) = y$.

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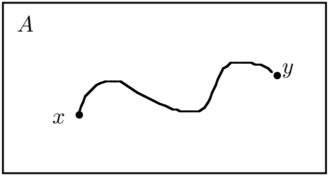
Although a path is formally a map $p : [0, 1] \to A$, we picture it as the image $p([0, 1]) \subseteq A$.



Path Connectedness

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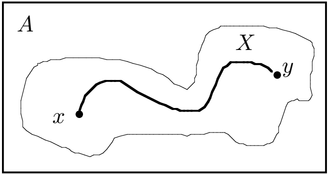

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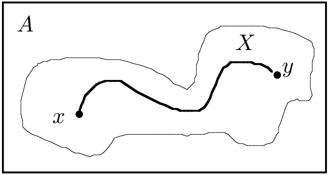

- Path connectedness is very useful (crucial in geometric and algebraic topology).

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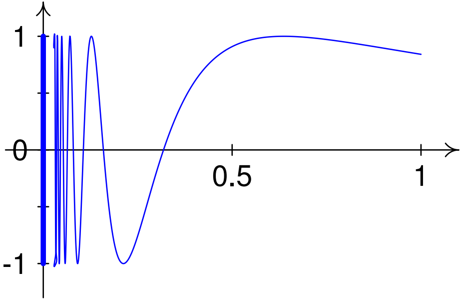
- Following subspace of $\mathbb{R}^2$ is connected but not path connected.



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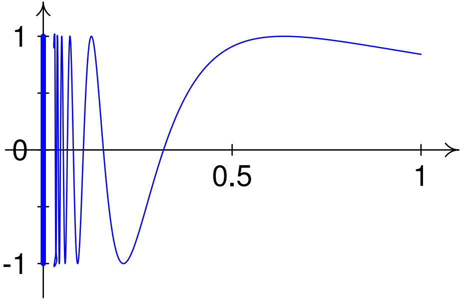

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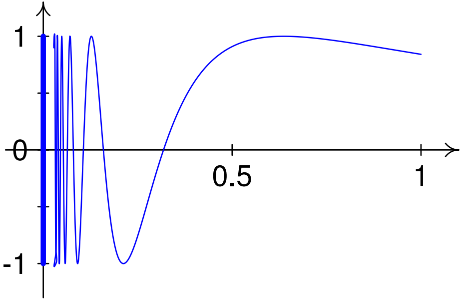

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This often makes it easier to prove connectedness.

Keeping Things Together

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Theorem 4 (Image of a Connected Space)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces and let $f : A \to B$ be continuous. If $(A, \mathcal{S})$ is connected, then $f(A)$ is connected.

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Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces and let $f : A \to B$ be continuous. If $Z \subseteq A$ is connected, then so is $f(Z)$.

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Proof: Use Theorem 4 and fact that $f|_Z$ is continuous by Theorem 4 of Lecture 4.

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Theorem 5 (Intermediate Value Theorem)

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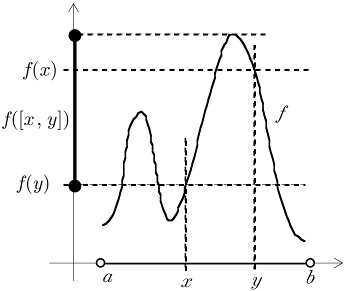
Often, $\text{dom}(f)$ is a subset of $\mathbb{R}$ and X is an interval.



Path Connectedness and Continuity

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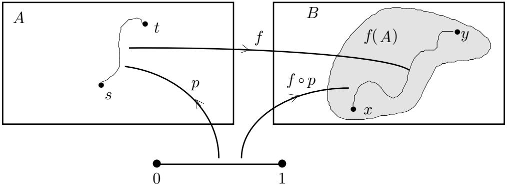

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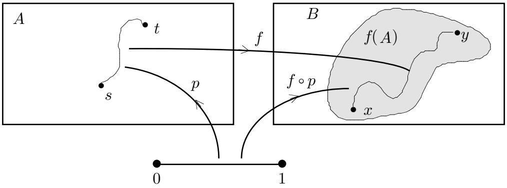

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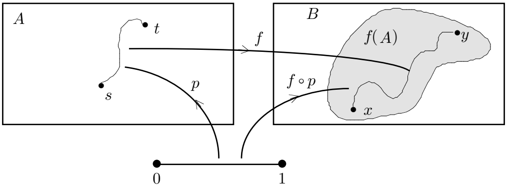

Corollary 2 (Image of Path Connected Set)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces and let $f : A \to B$ be continuous. If $Z \subseteq A$ is path connected, then so is $f(Z)$.

Proof: Use theorem and fact that $f|_Z$ is continuous by Theorem 4 of Lecture 4.