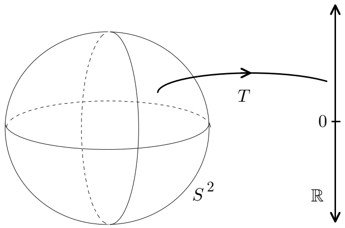
Outline

Good Things Come in Small Packages

Consider the two dimensional sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a model of the surface of the earth:



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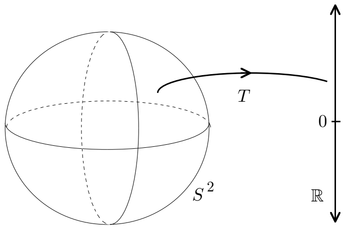

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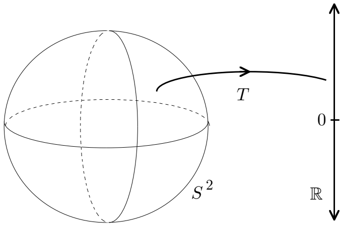

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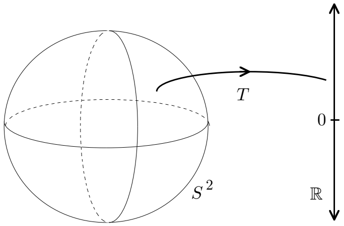

If $T : S^2 \to \mathbb{R}$ measures temperature at some instant in time we expect:

- T is continuous.
- T takes maximum and minimum values.

What property of S^2 convinces us that the maximum and minimum must exist?

Consider the Possibilities

- Not all continuous maps have maxima and minima, even on simple domains. EG:

$$f : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto \frac{1}{x}$$

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  - If $X = (1, 2]$ then $f(X)$ is bounded but has no maximum element.
- So there are many possibilities!

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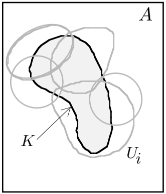
- The right topological notion is the concept of compactness.
- The definition of compactness is ingenious, and quite sophisticated.
- Some motivation for the definition of compactness is given in Chapter 5.1 of the printed notes.

Covers

Definition 1 (Cover)

Let K be a set. A set of sets $\{U_i : i \in I\}$ is a cover for K if $K \subset \bigcup_{i \in I} U_i$.

The term “cover" is suggested by what happens in one or two dimensions.



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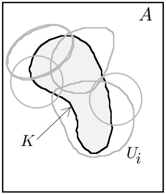

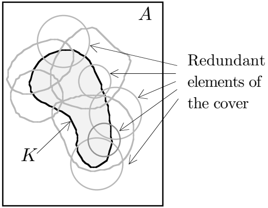

We can often throw away some sets in a cover for K and still have a cover for K.

Subcovers and Open Covers

Definition 2 (Subcover)

Let $\{U_i : i \in I\}$ cover K and suppose $\{U_j : j \in J\}$ also covers K for some $J \subset I$. We call $\{U_j : j \in J\}$ is a subcover of $\{U_i : i \in I\}$.

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An open cover is one in which all sets are open. A cover is finite if it contains a finite number of sets.

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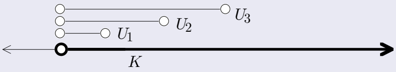
- The covering sets themselves need not be finite.

Consider the Possibilities

Example 1 ($K = \mathbb{R}^+,\ U_n = (0, n),\ I = \mathbb{N}$)

- $\{U_n : n \in \mathbb{N}\}$ an open cover for K.
- $\{U_n : n \in 2\mathbb{N}\}$ is a subcover.
- There is no finite subcover.



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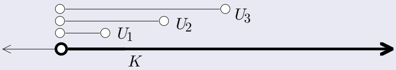
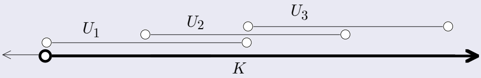


Example 2 ($K = \mathbb{R}^+,\ U_n = (n-1, n+1),\ I = \mathbb{N}$.)

- $\{U_n : n \in \mathbb{N}\}$ an open cover for K.
- Contains no proper subcover (prove it!)



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Example 3 ($K = (1, 3)$, $U_n = (n-1, n+1)$, $I = \mathbb{N}$)

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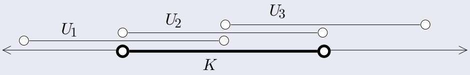

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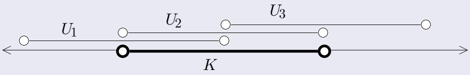
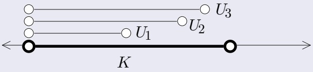


Example 4 ($K = (1, 3)$, $U_n = (1, 3 - \frac{1}{n})$, $I = \mathbb{N}$)

- $\{U_n : n \in \mathbb{N}\}$ an open cover for K.
- It has no finite subcover.



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- There are many equivalent definitions.
- They typically apply only in restricted settings.

Compact Intervals

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A bounded closed interval $[a, b]$ *in* $\mathbb{R}$ *is compact.*

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Sketch of Proof (This technique is called a “creeping argument")

Let $\mathcal{C} = \{U_i : i \in I\}$ be an open cover of $[a, b]$. Let $S = \{x \in [a, b] : \exists \text{ a finite subcover of } \mathcal{C} \text{ for } [a, x]\}$. We show $b \in S$.

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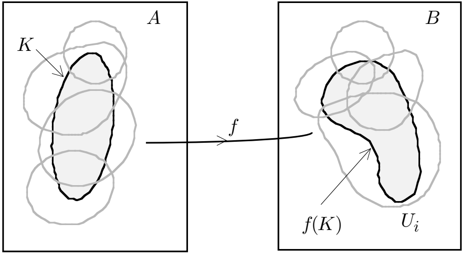
6. For $x \in S$ with $x < b$, find a point $x < y \leq b$ such that the same finite subcover covers $[a, y]$. Hence $y \in S$.
7. Use (1), (2) and (3) to show that $z = b$, so $b \in S$.

Proving sets are Compact

Theorem 1 (Continuous Image of a Compact Set)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces. If $f : A \to B$ is continuous and K is compact in A, then $f(K)$ is compact in B.



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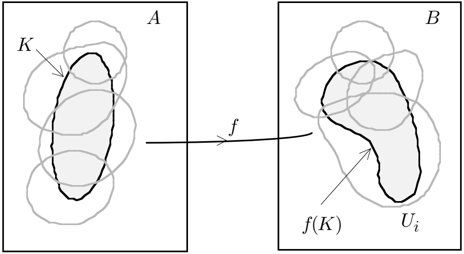

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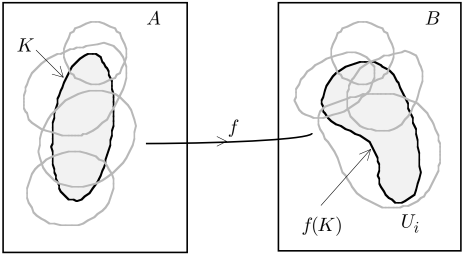
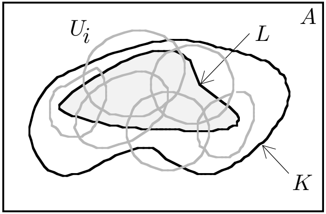


- Proof: the preimage of an open cover is an open cover.
- We can then use compact subsets of the domain of continuous maps to obtain compact subsets in the codomain.

Closed Subsets

Theorem 2 (Closed subsets of Compact Sets)

Let $(A, \mathcal{S})$ be a topological space, let K be a compact subset of A and let L be a subset of K which is closed in A. Then L is compact.



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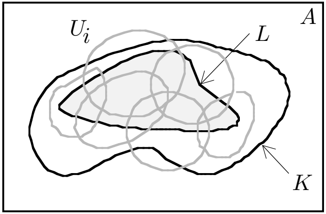

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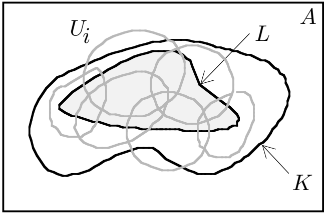

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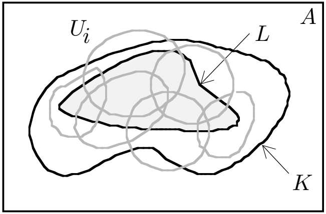

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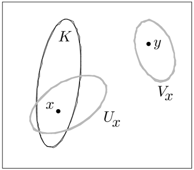
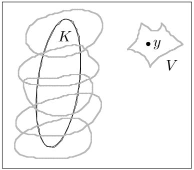
- Proof: To a cover of L add the open set $A \setminus L$.

Compact and Hausdorff

Theorem 3 (Compactness in Hausdorff Spaces)

A compact subset K of a Hausdorff space $(A, \mathcal{S})$ is closed.





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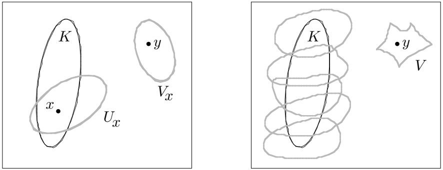

Proof: let $y \notin K$. For every $x \in K$ take disjoint $U_x \supset x$ and $V_x \supset y$. Take finite subcover of K by $\{U_x\}$ and the intersection of the corresponding V_x. It is open and disjoint from K. Use

Compact and Hausdorff

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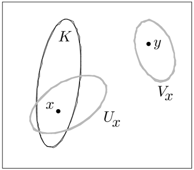

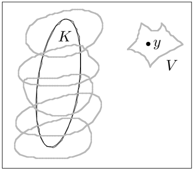

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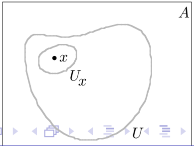
Lemma 2

Let $(A, \mathcal{S})$ be a topological space. Then U is open iff for all $x \in U$ there is an open set U_x such that $x \in U_x \subset U$.



Theorem 4 (Compact in a Metric Space)

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In an arbitrary metric space, the converse may not be true, but it is true in $\mathbb{R}^n$.

Noncompact unit closed ball

Example 5

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The complete characterisation of compact subsets of $\mathbb{R}^n$ is given by the celebrated Heine-Borel Theorem.

Heine-Borel

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If $n > 1$, given an open cover of the cube $C = C_1$ with no finite subcover, divide C_1 into 2^n smaller cubes; at least one of them, C_2, needs infinite cover; divide it into 2^n smaller cubes and so on. We get a sequence C_m of nested cubes, neither of which has a finite subcover.

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Proof.

By Theorem 1, $f(K)$ is compact in $\mathbb{R}$. By Theorem 4, $f(K)$ is closed and bounded. Let $b = \sup f(K)$. Then $b \in f(K)$, so $b = \max f(K)$ and there exists $x \in K$ such that $f(x) = b$. The proof for the minimum is similar. □

Corollary 1 (Weierstrass)

A continuous function $f : [a, b] \to \mathbb{R}$ attains its maximum and minimum.