Outline

Closed sets

- To motivate the definition of open sets in a metric spaces in Lecture 2, we used the idea of “boundary".

Closed sets

- To motivate the definition of open sets in a metric spaces in Lecture 2, we used the idea of “boundary".
- We think of an open set as one that doesn’t contain any of its boundary points.

Closed sets

- To motivate the definition of open sets in a metric spaces in Lecture 2, we used the idea of “boundary".
- We think of an open set as one that doesn’t contain any of its boundary points.
- We think of a closed set as one containing all its boundary points.

Closed sets

- To motivate the definition of open sets in a metric spaces in Lecture 2, we used the idea of “boundary".
- We think of an open set as one that doesn’t contain any of its boundary points.
- We think of a closed set as one containing all its boundary points.
- If B is not closed, we could obtain a closed set $\overline{B}$ by adding any missing boundary points.

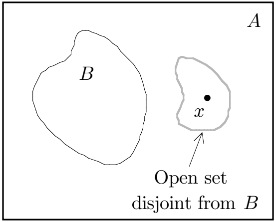
Closed sets

- To motivate the definition of open sets in a metric spaces in Lecture 2, we used the idea of “boundary".
- We think of an open set as one that doesn't contain any of its boundary points.
- We think of a closed set as one containing all its boundary points.
- If B is not closed, we could obtain a closed set $\overline{B}$ by adding any missing boundary points.
- Closed sets are very useful, as Theorems of Lecture 6 show.

Closed sets

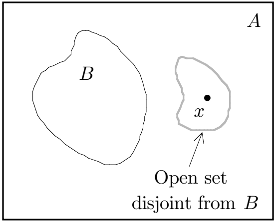
- To motivate the definition of open sets in a metric spaces in Lecture 2, we used the idea of “boundary".
- We think of an open set as one that doesn’t contain any of its boundary points.
- We think of a closed set as one containing all its boundary points.
- If B is not closed, we could obtain a closed set $\overline{B}$ by adding any missing boundary points.
- Closed sets are very useful, as Theorems of Lecture 6 show.
- Thus, given set B that may not be closed, it is often convenient to find a closed set $\overline{B} \supset B$ that is as small as possible.

Motivating the Definition



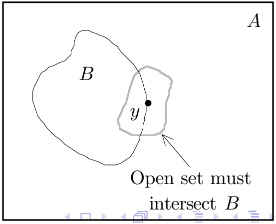
If x is “too far" from B to be a boundary point there should be an open set $U \ni x$ such that $U \cap B = \varnothing$.

Motivating the Definition



If x is “too far" from B to be a boundary point there should be an open set $U \ni x$ such that $U \cap B = \varnothing$.

If y is a boundary point of B, we feel that any open set $U \ni y$ must intersect B.



Definition of Closure

Definition 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. A point x is a point of closure of B, if any open set containing x intersects B.
The set $\overline{B}$ of all points of closure of B is called the closure of B.

Definition of Closure

Definition 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. A point x is a point of closure of B, if any open set containing x intersects B.
The set $\overline{B}$ of all points of closure of B is called the closure of B.

Theorem 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\overline{B}$ is a closed set containing B.

Definition of Closure

Definition 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. A point x is a point of closure of B, if any open set containing x intersects B.
The set $\overline{B}$ of all points of closure of B is called the closure of B.

Theorem 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\overline{B}$ is a closed set containing B.

Proof: $\overline{B} \supset B$ by definition.

Definition of Closure

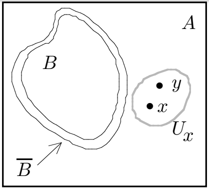
Definition 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. A point x is a point of closure of B, if any open set containing x intersects B.
The set $\overline{B}$ of all points of closure of B is called the closure of B.

Theorem 1 (Closure)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\overline{B}$ is a closed set containing B.

Proof: $\overline{B} \supset B$ by definition.
To prove that $\overline{B}$ is closed, take any $x \in (\overline{B})'$. Then there exists an open $U_x \ni x$ such that $U_x \cap B = \varnothing$. If $y \in U_x \cap \overline{B}$, then any open set containing y has a nonempty intersection with B. In particular, U_x – a contradiction. So for any $x \in (\overline{B})'$, there is an open $U_x \ni x$ such that $U_x \subset (\overline{B})'$. Lemma 2 of Lecture 6 then implies that $(\overline{B})'$ is open, and hence $\overline{B}$ is closed.



Small Target Strategy

- We feel that $\overline{B}$ must be the "smallest" closed set containing B, but what can this mean?

Small Target Strategy

- We feel that $\overline{B}$ must be the "smallest" closed set containing B, but what can this mean?
- Let K be the intersection of all closed sets containing B, so K is closed by Theorem 1 of Lecture 3.

Small Target Strategy

- We feel that $\overline{B}$ must be the "smallest" closed set containing B, but what can this mean?
- Let K be the intersection of all closed sets containing B, so K is closed by Theorem 1 of Lecture 3.
- Since A is closed and $B \subset A$, $K \neq \varnothing$.

Small Target Strategy

- We feel that $\overline{B}$ must be the "smallest" closed set containing B, but what can this mean?
- Let K be the intersection of all closed sets containing B, so K is closed by Theorem 1 of Lecture 3.
- Since A is closed and $B \subset A$, $K \neq \varnothing$.
- K is the "smallest" closed set containing B.

Theorem 2 (Smallness of Closure)

If $(A, \mathcal{S})$ is a topological space and $B \subset A$ then $\overline{B}$ is the intersection of all the closed sets in A which contain B.

Proof: $\overline{B}$ is closed and contains B, so $\overline{B} \supset K$.

Small Target Strategy

- We feel that $\overline{B}$ must be the "smallest" closed set containing B, but what can this mean?
- Let K be the intersection of all closed sets containing B, so K is closed by Theorem 1 of Lecture 3.
- Since A is closed and $B \subset A$, $K \neq \varnothing$.
- K is the "smallest" closed set containing B.

Theorem 2 (Smallness of Closure)

If $(A, \mathcal{S})$ is a topological space and $B \subset A$ then $\overline{B}$ is the intersection of all the closed sets in A which contain B.

Proof: $\overline{B}$ is closed and contains B, so $\overline{B} \supset K$.
If $\overline{B} \neq K$, then there is $x \in \overline{B} \setminus K$, so that there is a closed set $L \supset B$ which doesn't contain x. But then $L' \ni x$, is open and disjoint from B. Contradiction with $x \in \overline{B}$.

Some Examples

Example 1

$\overline{(0,1)} = [0,1]$ and $\overline{\mathbb{Q}} = \mathbb{R}$.

Some Examples

Example 1

$\overline{(0,1)} = [0,1]$ and $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 2

$\overline{\mathbb{Q}^n} = \mathbb{R}^n$.

Some Examples

Example 1

$\overline{(0,1)} = [0,1]$ and $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 2

$\overline{\mathbb{Q}^n} = \mathbb{R}^n$.

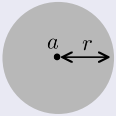
Example 3

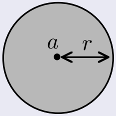
In $\mathbb{R}$ with the discrete topology $\overline{\mathbb{Q}} = \mathbb{Q} \neq \mathbb{R}$ because all sets are both open and closed. In fact $B = \overline{B}$ for every $B \subset \mathbb{R}$.

A Word of Warning

Example 4

In $\mathbb{R}^2$ we have $\overline{B_r(a)} = \overline{B}_r(a)$.
This is not true in all metric spaces.

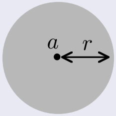


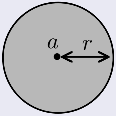


A Word of Warning

Example 4

In $\mathbb{R}^2$ we have $\overline{B_r(a)} = \overline{B}_r(a)$.
This is not true in all metric spaces.





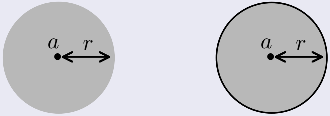
Example 5

Let $A = [\mathbb{R}^2 \setminus B_1((0,0))] \cup \{(0,0)\}$

A Word of Warning

Example 4

In $\mathbb{R}^2$ we have $\overline{B_r(a)} = \overline{B}_r(a)$.
This is not true in all metric spaces.



Example 5

Let $A = [\mathbb{R}^2 \setminus B_1((0,0))] \cup \{(0,0)\}$

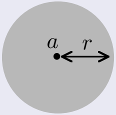
In this space

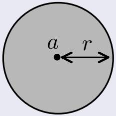
- $\overline{B}_1((0,0)) = S^1 \cup \{(0,0)\}$

A Word of Warning

Example 4

In $\mathbb{R}^2$ we have $\overline{B_r(a)} = \overline{B}_r(a)$.
This is not true in all metric spaces.



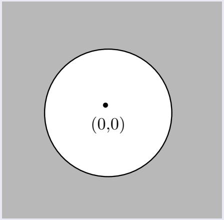


Example 5

Let $A = [\mathbb{R}^2 \setminus B_1((0,0))] \cup \{(0,0)\}$

In this space

- $\overline{B}_1((0,0)) = S^1 \cup \{(0,0)\}$
- $\overline{B_1((0,0))} = B_1((0,0)) = \{(0,0)\}$.



Dense and Denser

- The idea that B is dense in A means that wherever we look in A we find elements of B.

Dense and Denser

- The idea that B is dense in A means that wherever we look in A we find elements of B.
- We make this vague idea precise by saying “whichever non-empty open set we look in":

Definition 2 (Dense Subset)

We call D a dense subset of a topological space $(A, \mathcal{S})$, if $D \cap U \neq \varnothing$ for every non-empty open set U.

Dense and Denser

- The idea that B is dense in A means that wherever we look in A we find elements of B.
- We make this vague idea precise by saying "whichever non-empty open set we look in":

Definition 2 (Dense Subset)

We call D a dense subset of a topological space $(A, \mathcal{S})$, if $D \cap U \neq \varnothing$ for every non-empty open set U.

- We have seen that $(0, 1)$ is dense in $[0, 1]$ and $\mathbb{Q}$ is dense in $\mathbb{R}$.

Dense and Denser

- The idea that B is dense in A means that wherever we look in A we find elements of B.
- We make this vague idea precise by saying “whichever non-empty open set we look in":

Definition 2 (Dense Subset)

We call D a dense subset of a topological space $(A, \mathcal{S})$, if $D \cap U \neq \varnothing$ for every non-empty open set U.

- We have seen that $(0, 1)$ is dense in $[0, 1]$ and $\mathbb{Q}$ is dense in $\mathbb{R}$.
- However $\mathbb{Q}$ is not dense in $\mathbb{R}$ with the discrete topology.

Dense and Denser

- The idea that B is dense in A means that wherever we look in A we find elements of B.
- We make this vague idea precise by saying “whichever non-empty open set we look in":

Definition 2 (Dense Subset)

We call D a dense subset of a topological space $(A, \mathcal{S})$, if $D \cap U \neq \varnothing$ for every non-empty open set U.

- We have seen that $(0, 1)$ is dense in $[0, 1]$ and $\mathbb{Q}$ is dense in $\mathbb{R}$.
- However $\mathbb{Q}$ is not dense in $\mathbb{R}$ with the discrete topology.

Theorem 3 (Density and Closure)

Let $(A, \mathcal{S})$ be a topological space. Then $D \subset A$ is dense in A iff $\overline{D} = A$.

Boundary and Interior

We can need to fulfil a promise. Back in Lecture 2, we said we would give a formal definition of "boundary".

Definition 3 (Boundary)

In a topological space $(A, \mathcal{S})$, the *boundary* of a set B is $\partial B = \overline{B} \cap \overline{B'}$; that is, the intersection of the closures of B and its complement.

Boundary and Interior

We can need to fulfil a promise. Back in Lecture 2, we said we would give a formal definition of “boundary”.

Definition 3 (Boundary)

In a topological space $(A, \mathcal{S})$, the *boundary* of a set B is $\partial B = \overline{B} \cap \overline{B'}$; that is, the intersection of the closures of B and its complement.

Another important, related notion is that of the interior.

Definition 4 (Interior)

In a topological space $(A, \mathcal{S})$, the *interior* of a set B is the set B^0 of points $x \in B$ for which there exists an open set $U \in \mathcal{S}$ with $x \in U \subset B$.

Boundary in terms of closure and interior

Theorem 4 (Boundary and Interior)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\partial B = \overline{B} \backslash B^0$.

Boundary in terms of closure and interior

Theorem 4 (Boundary and Interior)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\partial B = \overline{B} \backslash B^0$.

Proof.

Let $x \in A$. Since $\partial B = \overline{B} \cap \overline{(A \backslash B)}$, we have

$$x \in \partial B \quad \equiv \quad x \in \overline{B} \text{ and } x \in \overline{(A \backslash B)}$$

Boundary in terms of closure and interior

Theorem 4 (Boundary and Interior)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\partial B = \overline{B} \backslash B^0$.

Proof.

Let $x \in A$. Since $\partial B = \overline{B} \cap \overline{(A \backslash B)}$, we have

$$x \in \partial B \quad \equiv \quad x \in \overline{B} \text{ and } x \in \overline{(A \backslash B)}$$

$$\equiv \quad x \in \overline{B} \quad \text{and for every open set } U \text{ with } x \in U, \text{ we have}$$

$$U \cap A \backslash B \neq \varnothing$$

Boundary in terms of closure and interior

Theorem 4 (Boundary and Interior)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\partial B = \overline{B} \backslash B^0$.

Proof.

Let $x \in A$. Since $\partial B = \overline{B} \cap \overline{(A \backslash B)}$, we have

$$\begin{aligned}
x \in \partial B \quad &\equiv \quad x \in \overline{B} \text{ and } x \in \overline{(A \backslash B)} \\
&\equiv \quad x \in \overline{B} \quad \text{and for every open set } U \text{ with } x \in U, \text{ we have} \\
&\qquad\qquad U \cap A \backslash B \neq \varnothing \\
&\equiv \quad x \in \overline{B} \quad \text{and for no open set } U \text{ with } x \in U, \text{ do we have } U \subset B
\end{aligned}$$

Boundary in terms of closure and interior

Theorem 4 (Boundary and Interior)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\partial B = \overline{B} \backslash B^0$.

Proof.

Let $x \in A$. Since $\partial B = \overline{B} \cap \overline{(A \backslash B)}$, we have

$$\begin{aligned}
x \in \partial B \quad &\equiv \quad x \in \overline{B} \text{ and } x \in \overline{(A \backslash B)} \\
&\equiv \quad x \in \overline{B} \quad \text{and for every open set } U \text{ with } x \in U, \text{ we have} \\
&\qquad\qquad U \cap A \backslash B \neq \varnothing \\
&\equiv \quad x \in \overline{B} \quad \text{and for no open set } U \text{ with } x \in U, \text{ do we have } U \subset B \\
&\equiv \quad x \in \overline{B} \quad \text{and } x \notin B^0
\end{aligned}$$

Boundary in terms of closure and interior

Theorem 4 (Boundary and Interior)

Let $(A, \mathcal{S})$ be a topological space and let $B \subset A$. Then $\partial B = \overline{B} \backslash B^0$.

Proof.

Let $x \in A$. Since $\partial B = \overline{B} \cap \overline{(A \backslash B)}$, we have

$$
\begin{aligned}
x \in \partial B \quad &\equiv \quad x \in \overline{B} \text{ and } x \in \overline{(A \backslash B)} \\
&\equiv \quad x \in \overline{B} \quad \text{and for every open set } U \text{ with } x \in U, \text{ we have} \\
&\qquad\qquad U \cap A \backslash B \neq \varnothing \\
&\equiv \quad x \in \overline{B} \quad \text{and for no open set } U \text{ with } x \in U, \text{ do we have } U \subset B \\
&\equiv \quad x \in \overline{B} \quad \text{and } x \notin B^0 \\
&\equiv \quad x \in \overline{B} \backslash B^0.
\end{aligned}
$$

Hence $\partial B = \overline{B} \backslash B^0$. □