Outline

Divide and Conquer

Consider a topological space $(A, \mathcal{S})$, a set B and a map $f : A \to B$. Recall from Definition 4 of Lecture 4 that the set $\mathcal{T}_f = \{U \subset B \,:\, f^{-1}(U) \in \mathcal{S}\}$ is a topology on B, called the final topology, which makes f continuous.

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Definition 1 (Quotient Topology)

Let $(A, \mathcal{S})$ be a topological space, let B be a set, and let $\pi : A \to B$ be an onto map. The final topology $\mathcal{T}_\pi$ is called the quotient topology on B. The space $(B, \mathcal{T}_\pi)$ is called a quotient of $(A, \mathcal{S})$, and when there is no possible confusion, B is called a quotient of A. The map π is called the quotient map (it is sometimes also called the natural projection map).

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Example 1

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Example 2

The map $f : \mathbb{R} \to S^1 : x \mapsto e^{2\pi x i}$ is a quotient map.

Ready to collapse?

Quotients “collapse" chunks of a topological space. We need a way of saying which chunks. Recall from Lecture 1:

Definition 2 (Partition)

A set $\mathcal{B}$ of non-empty subsets of A is called a partition of S provided

1. $(\forall x \in A)(\exists B \in \mathcal{B}) \; : \; x \in B$
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Definition 3 (Natural Projection)

Let $\mathcal{B}$ be a partition of A. The unique map $\pi : A \to \mathcal{B}$ such that $x \in \pi(x)$ is called the natural projection of A onto $\mathcal{B}$.

Partitions and equivalences

Conversely, if one is given an onto map $\pi : A \to B$ to some set B, then π determines a partition $\mathcal{P}$ of A:

$$\mathcal{P} := \{\pi^{-1}(\{x\}) : x \in B\}.$$

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Given an equivalence relation and $x \in A$, the equivalence class $[x]$ is the set $[x] := \{y \in A : y \sim x\}$. Clearly, the set of equivalence classes forms a partition of A. Conversely, given a partition $\mathcal{P}$ of A, there is a corresponding equivalence relation: we set $x \sim y$, if x and y belong to the same part of $\mathcal{P}$.

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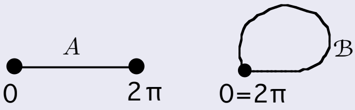
Let $A = [0, 2\pi]$



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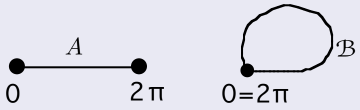

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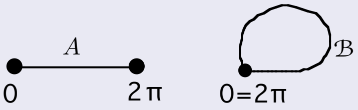

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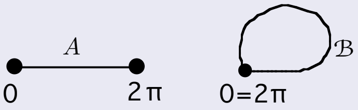

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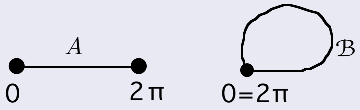

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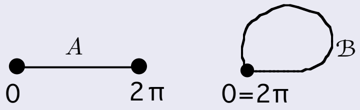

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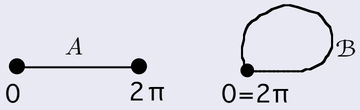

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This shortcut is often used to define quotients:

- State which points do get identified.
- Assume other points are not identified.

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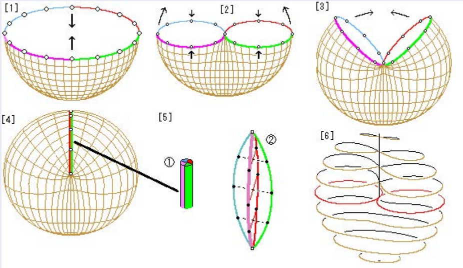

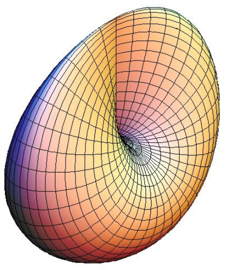

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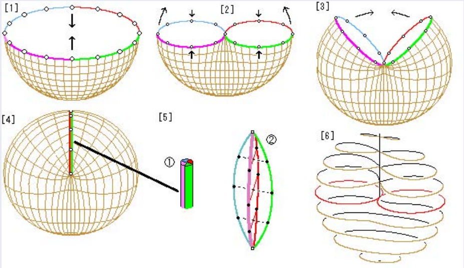

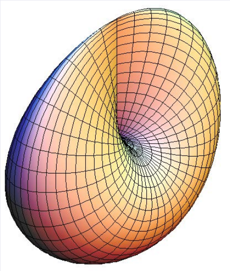

Exercise: What do we get if we identify the bounding circles of two discs? of a Möbius band and a disc? of two Möbius bands?

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Proof: as U is saturated, $\pi^{-1}(\pi(U)) = U$.

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Let $\mathcal{B}$ be a quotient of a topological space $(A, \mathcal{S})$ and let $\pi : A \to \mathcal{B}$ denote the natural projection. We say that a subset $U \subset A$ is π-saturated if $\pi^{-1}(\{z\}) \subset U$ for all $z \in \pi(U)$.

Theorem 2 (π-saturated open sets)

For all open π-saturated sets $U \subset A$, the image $\pi(U)$ is open in $\mathcal{B}$.

Proof: as U is saturated, $\pi^{-1}(\pi(U)) = U$.

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Let $\mathcal{B}$ be a quotient of a topological space $(A, \mathcal{S})$. If $\mathcal{B}$ is Hausdorff, then for each $x \in \mathcal{B}$, $\pi^{-1}(x)$ is closed in A.

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Proof: x' is open in $\mathcal{B}$ by Hausdorffity, hence $\pi^{-1}(x')$ is open in A, as π is continuous. Then $(\pi^{-1}(x'))'$ is closed in A. But $(\pi^{-1}(x'))' = \pi^{-1}(x)$.

Some Results...

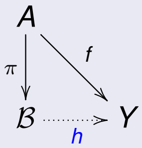
And a final result

Lemma 1 (sort of universality)

Let $\mathcal{B}$ be a quotient of a space $(A, \mathcal{S})$, let $(Y, \mathcal{T})$ be a space and let $f : A \to Y$ be a continuous map satisfying

$$\pi(x) = \pi(y) \;\Rightarrow\; f(x) = f(y).$$

Then $h : \mathcal{B} \to Y : \pi(x) \mapsto f(x)$ is well defined and continuous. If f is onto, so is h.



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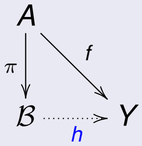

Proof: by definition (final topology is universal, all that).