MAT4TOP **ASSIGNMENT 1, 2015**

1. Return your assignment solutions to the MAT4TOP assignment box on level 3, PS2, by 2pm on August 12th.
2. Please start with the following statement of originality, which must be signed by you: `"This is my own work. I have not copied any of it from anyone else."`
3. The total of this assignment is 14 marks (not counting the Bonus Questions). All the questions are marked.

1. (Set Cardinality and Cantor's Theorem)

Let X and Y be sets. We say that

1. $|X| \leq |Y|$, if there is an *one-to-one* map $\psi : X \to Y$.
2. $|X| = |Y|$, if $|X| \leq |Y|$ and $|Y| \leq |X|$.
3. $|X| < |Y|$ if $|X| \leq |Y|$ is true, but $|X| = |Y|$ is false.

Here we prove that $|X| < |\mathcal{P}(X)|$ (Cantor's Theorem).

(a) Show that $|X| \leq |\mathcal{P}(X)|$ by defining an one-to-one mapping $\psi : X \to \mathcal{P}(X)$.

(b) (Barber paradox). Suppose that $|X| = |\mathcal{P}(X)|$ so there is a one-to-one and onto map $\psi : X \to \mathcal{P}(X)$. Let
$$A = \{x \in X : x \notin \psi(x)\}.$$
Since ψ is onto, $\psi(z) = A$ for some $z \in X$ and either $z \in A$ or $z \notin A$. Show that both of these assumptions lead to a contradiction. What do you conclude?

2. (Topologies) Prove that

(a) the Finite Closed Topology $\mathcal{F}$ (Definition 8 of Lecture 3)

(b) the Initial Topology $\mathcal{I}_f$ (Definition 11 of Lecture 3)

are indeed topologies, by checking that they satisfy the axioms (T1), (T2) and (T3).

3. (Continuity) Prove that

(a) a map is continuous if and only if preimages of all closed sets are closed.

(b) For any continuous function $f : \mathbb{R} \to \mathbb{R}$ (in the Euclidean topology), the set of solutions of the equation $f(x) = 0$ is closed in $\mathbb{R}$.

(c) **(Bonus Question, 3 marks, "the converse" to (b))** any closed subset in $\mathbb{R}$ is the set of solutions of the equation $f(x) = 0$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$.

This could be sort of hardish. One possible construction uses the following fact (which needs to be proved first): any (Euclidean) open subset of $\mathbb{R}$ is a countable disjoint union of open intervals (a set I is called *countable* if $|I| = |\mathbb{N}|$).

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Comment. The statement in (c) holds in much more general settings: any closed subset of $\mathbb{R}^n$ is the zero set of a C^∞ map from $\mathbb{R}^n$ to $\mathbb{R}^m$. But "C^∞" here cannot be replaced by analyticity, which is an evidence of a deep and fundamental difference between real differentiability and real analyticity.

4. (Theorem) Here we prove Theorem 1 of Lecture 4. Let A and B be topological spaces and $f : A \to B$ a map.

(a) Show that for all $U \subset A$ and $V \subset B$

$$f(U) \subset V \implies U \subset f^{-1}(V).$$

(b) Suppose f is continuous in the sense of Definition 9 of Lecture 3 and V is open in B. Use the result of part (a) to show that for each $x \in f^{-1}(V)$ there is an open subset U_x of A such that

$$x \in U_x \subset f^{-1}(V).$$

(c) Use the result of part (b) and property (T2) of open sets to complete the proof that $f^{-1}(V)$ is open.

(d) Use Lemma 4(5) of Lecture 1 to show that if $f^{-1}(V)$ is open for every open subset $V$of B then f is continuous in the sense of Definition 9 of Lecture 3.

Bonus questions (3 marks)

Find all the metric spaces $(A, \mathcal{S})$ having the property that every subset of A is either open or closed.

If you have time, do the same for topological spaces (no marks for that, but please show me your solution. One such example would be $A = \{a, b\}$, $\mathcal{S} = \{\varnothing, \{a\}, A\}$).

Hint: (not really).