A

1. (3 marks)

(a) $\psi : X \to \mathcal{P}(X) : x \mapsto \{x\}$ is one-to-one since $\psi(x) = \psi(y) \implies \{x\} = \{y\} \implies x = y$.

(b) If $z \in A$ then $z \notin \psi(z) = A$, which is a contradiction. If $z \notin A$ then $z \in \psi(z) = A$ which is also a contradiction. Since both cases lead to a contradiction, we conclude that there can be no such one-to-one and onto map $X \to \mathcal{P}(X)$ so $|X| < |\mathcal{P}(X)|$.

2. (4 marks)

(a)(T1) $\varnothing \in \mathcal{F}$ by definition and $A \setminus A = \varnothing$ is finite so $A \in \mathcal{F}$.

(T2) First note that if all the U_i are empty, then $\cup_{i\in I} U_i = \varnothing$ and so $\cup_{i\in I} U_i \in \mathcal{F}$. If one of the U_i is nonempty, say U_j, then $A \setminus U_j$ is finite, so by de Morgan's laws

$$A \setminus \left(\bigcup_{i\in I} U_i \right) = \bigcap_{i\in I} (A \setminus U_i) \subset A \setminus U_j.$$

Hence $A \setminus (\cup_{i\in I} U_i)$ is finite and thus $\cup_{i\in I} U_i \in \mathcal{F}$.

(T3) Let $U, V \in \mathcal{F}$. If $U \cap V = \varnothing$, then $U \cap V \in \mathcal{F}$. If $U \cap V \neq \varnothing$, then $U \neq \varnothing$ and $V \neq \varnothing$ so $A \setminus U$ and $A \setminus V$ are finite and $A \setminus (U \cap V) = (A \setminus U) \cup (A \setminus V)$ is finite, giving $U \cap V \in \mathcal{F}$.

(b)(T1) $\varnothing = f^{-1}(\varnothing) \in \mathcal{I}_f$ and $B = f^{-1}(A) \in I_f$.

(T2) Suppose U_i, $i \in I$, are open in B and $U_i = f^{-1}(V_i)$, where V_i are open in A. Then by Lemma 3(1) of Lecture 1, $\cup_{i\in I} U_i = \cup_{i\in I} f^{-1}(V_i) = f^{-1}(\cup_{i\in I} V_i) \in \mathcal{I}_f$.

(T3) Let $U_1, U_2 \in \mathcal{I}_f$, with $U_i = f^{-1}(V_i)$, $i = 1, 2$, where V_1, V_2 are open in A. Then by Lemma 3(2) of Lecture 1, $U_1 \cup U_2 = f^{-1}(V_1) \cup f^{-1}(V_2) = f^{-1}(V_1 \cup V_2) \in \mathcal{I}_f$.

3. (3 marks)

(a) Let $f : A \to B$ be a map of topological spaces. Let C be a closed subset of B. Then $B \setminus C$ is open in B so $f^{-1}(B \setminus C)$ is open in A by the definition of continuity. Since $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$ by Lemma 4(7) of Lecture 1, $f^{-1}(C)$ is closed in A. The converse follows by reverting the arguments.

(b) Since the solution set of $f(x) = 0$ is precisely $f^{-1}(0)$, it is closed by the result of (a).

(c) **Bonus questions (3 marks)** some proof

4. (4 marks)

(a) Assume $f(U) \subset V$. Then $x \in U \implies f(x) \in V \implies x \in f^{-1}(V)$.

(b) Assume f is continuous, V is open in B and $x \in f^{-1}(V)$. Then $f(x) \in V$ so by the definition of continuity, there is an open set U_x of A such that $x \in U_x$ and $f(U_x) \subset V$. By part (a) $U_x \subset f^{-1}(V)$.

(c) Let $U = f^{-1}(V)$. For all $z \in U$ we have $z \in U_z$ so $z \in \bigcup_{x\in U} U_x$. Thus $U \subset \bigcup_{x\in U} U_x$. On the other hand, for all $y \in \bigcup_{x\in U} U_x$ there is an $x \in U$ such that $y \in U_x \subset U$. Hence $y \in U$. So $\bigcup_{x\in U} U_x \subset U$. Therefore, $\bigcup_{x\in U} U_x = U$. As the sets U_x are open, $\bigcup_{x\in U} U_x$ is open (by T2). So U is open.

(d) Suppose that $f^{-1}(V)$ is open in A for every open subset $V$of B. Let $a \in A$ and let V be an open subset of B such that $f(a) \in V$. By assumption $U = f^{-1}(V)$ is open in A and $a \in U$ since $f(a) \in V$. The result of Lemma 4(5) of Lecture 1 gives $f(f^{-1}(V)) \subset V$. Hence f is continuous in the sense of Definition 9 of Lecture 3.

Bonus questions (3 marks)
Some proof.