A

1. Return your assignment solutions to the MAT4TOP assignment box on level 3, PS2, by 2pm on August 26th.
2. Please start with the following statement of originality, which must be signed by you: `"This is my own work. I have not copied any of it from anyone else."`
3. The total of this assignment is 18 marks (not counting the Bonus Questions). All the questions are marked.

1. (Connectedness)

Suppose that X and Y are subsets of a topological space $(A, \mathcal{S})$ such that $X \cap Y \neq \varnothing$.

(a) Prove that if X and Y are connected, then $X \cup Y$ is connected.

(b) Prove that if X and Y are path connected, then $X \cup Y$ is path connected. Hint: use Theorem 5 of Lecture 4.

Comment: the construction in part (b) is called "the product of paths". We will need it later to define the fundamental group.

2. (Regularity)

A topological space $(A, \mathcal{S})$ is said to be *regular* if for all closed subsets $C \subset A$ and all $y \in A \setminus C$, there exist disjoint open sets U and V such that $y \in V$ and $C \subset U$. Prove that all compact Hausdorff spaces are regular.

3. (Dense and Hausdorff)

Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be topological spaces. Suppose that D is a dense subset of A, and that $f : A \to B$ is a continuous function that is constant on D.

(a) Show that if $(B, \mathcal{T})$ is Hausdorff, then f is constant on A.

(b) Give an example of spaces A, B, a dense subset $D \subset A$, and a continuous function $f : A \to B$ that is constant on D but is not constant on A.

4. (Compactness)

(a) Prove that any finite subset in a topological space is compact.

(b) Prove that a topological space with the finite closed topology is compact.

(c) Prove that any compact set in the Sorgenfrey topology on $\mathbb{R}$ is compact in the Euclidean topology (explain and use the fact that the Sorgenfrey topology is "finer" than Euclidean: any Euclidean open set is Sorgenfrey open).

(d) Prove that $[0, 1]$ is not Sorgenfrey compact.

(e) Prove that $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$ is Sorgenfrey compact.

(f) **(Bonus Question, 3 marks)** Prove that any Sorgenfrey compact set is countable (that is, has the same cardinality as $\mathbb{N}$). You can use the fact that the set of rational numbers is countable.

Bonus questions

1. (4 marks) A norm on a linear space defines a metric, which then defines a topology. Clearly, different norms give different metrics (think, e.g. of the Euclidean, the Manhattan, the max norms on the plane, etc.), but the following amazing fact is true: "all the norms on the finite-dimensional space $\mathbb{R}^n$ define the same topology" [1].

To prove that you can follow the plan below. Let e_i be a fixed basis for $\mathbb{R}^n$, and denote $\|\cdot\|_0$ the Euclidean norm on $\mathbb{R}^n$.

(a) Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Explain why for any $x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^n$, we have $\|x\| \leq \sum_{i=1}^n |x_i| \|e_i\|$.

(b) Deduce from (a) that there is a constant $C_2 > 0$ such that for all $x \in \mathbb{R}^n$ we have $\|x\| \leq C_2 \|x\|_0$.

(c) Deduce from (b) that the function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|$, is continuous (in the Euclidean topologies on $\mathbb{R}$ and $\mathbb{R}^n$).

(d) Explain why the restriction of $\|\cdot\|$ to the (Euclidean) unit sphere in $\mathbb{R}^n$ is continuous.

(e) Explain why there is a positive constant C_1 such that $C_1 \|x\|_0 \leq \|x\|$.

(f) Explain how (b) and (e) imply that the topologies defined by the norms $\|\cdot\|$ and $\|\cdot\|_0$ are the same and hence finish the proof.

2. (2 marks) Prove that there are two antipodal points on the equator with the same temperature (assume the temperature is continuous).

[1] In infinite dimensions, this is very untrue. Consider e.g. the space $C[0,1]$, with the sup norm and with the L_1 norm defined by $\|f\|_{L_1} = \int_0^1 |f(x)| dx$. You can prove that the sequence $f_n(x) = x^n$ converges to 0 in the L_1 norm, but not in the sup norm, and deduce from this that the resulting topologies are not the same.