MAT4TOP **Solutions to Assignment 2, 2015**

1. (6 marks)

(a) Let $\{C, D\}$ be a clopen partition for $X \cup Y$. We consider the various possibilities for the intersections between C, D, X and Y.

(i) If $X, Y \subset C$, then $X \cup Y \subset C$ and hence $D = \varnothing$, a contradiction. Similarly, we cannot have both $X \subset D$ and $Y \subset D$.

(ii) If $X \subset C$ and $Y \subset D$ (or vice versa), then $X \cap Y = \varnothing$, a contradicting our hypothesis.

(iii) The remaining possibilities are that at least one of X and Y must intersect both C and D. Suppose X intersects both C and D. Since C and D are open in $X \cup Y$, there are sets U and V open in A such that $C = U \cap (X \cup Y)$ and $D = V \cap (X \cup Y)$. Now $U \cap X$ and $V \cap X$ are

- both open in X, by definition,
- disjoint since $U \cap X \subset U \cap (X \cup Y) = C$ and $V \cap X \subset V \cap (X \cup Y) = D$,
- both non-empty since $\varnothing \neq C \cap X \subset U \cap X$ and $\varnothing \neq D \cap X \subset V \cap X$

and their union is X since they are both contained in X and $X \subset X \cup Y = C \cup D \subset U \cup V$. Hence $\{U \cap X, V \cap X\}$ is a clopen partition of X which shows that X is disconnected contradicting our hypothesis. Similarly, if Y intersected both C and D, it would be disconnected.

(b) Let $x, y \in X \cup Y$. If both x and y are in X (or both are in Y), there is nothing to prove. Otherwise suppose that $x \in X$ and $y \in Y$ (up to relabelling) and let $z \in X \cap Y$. Then there is a path p from x to z in X by the path connectedness of X and a path q from z to y in Y by the path connectedness of Y. The maps $t \mapsto p(2t)$ and $t \mapsto q(2t-1)$ are continuous on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ by Theorem 2 of Lecture 4. Theorem 5 of Lecture 4 now implies that

$$r : [0,1] \to A \ : \ t \mapsto \begin{cases} p(2t) & : \ t \in [0, \frac{1}{2}], \\ q(2t-1) & : \ t \in [\frac{1}{2}, 1]. \end{cases}$$

is continuous since $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in $[0, 1]$ and $p(1) = z = q(0)$. Since $r(0) = p(0) = x$ and $r(1) = q(1) = z$, we see that r is a path from x to z. The path r is called *the product* of the paths p and q.

2. (3 marks)

Let $C \subset A$ be closed and let $y \in A \setminus C$. As $(A, \mathcal{S})$ is compact, C is compact by Theorem 3 of Lecture 6. As $(A, \mathcal{S})$ is Hausdorff, for each $x \in C$ there exist disjoint open sets U_x and V_x such that $x \in U_x$ and $y \in V_x$. Since $\{U_x : x \in C\}$ is an open cover for C, compactness gives a finite $F \subset C$ such that $C \subset U = \bigcup_{x \in F} U_x$ and U is open by (T2).

The set $V = \bigcap_{x \in F} V_x$ is also open, and $y \in V$ since $y \in V_x$ for all x. Suppose $z \in V \cap U$. Since $z \in U = \bigcup_{x \in F} U_x$, we have $z \in U_w$ for some $w \in F$. Since $z \in V = \bigcap_{x \in F} V_x$ we have $z \in V_x$ for all $x \in F$. In particular, we have $z \in V_w$, a contradiction. We conclude that $V \cap U = \varnothing$ which completes the proof.

3. (4 marks)

(a) Assume that f is constant on D, so there exists $x \in B$ with $f(a) = x$, for all $a \in D$. Assume that f is not constant on A. So there exists $z \in A$ with $f(z) \neq x$. Since $(B, \mathcal{T})$ is Hausdorff, there exist disjoint open sets $U, V \subset B$ with $x \in U$, $f(z) \in V$. Since f is continuous, $f^{-1}(V)$ is open in A. As D is dense in A, there exists $y \in D \cap f^{-1}(V)$. Since $y \in f^{-1}(V)$, we have $f(y) \in V$. But $x \in U$ and $U \cap V = \varnothing$, so $x \notin V$. Hence $f(y) \neq x$. But since $y \in D$, we have $f(y) = x$. This is a contradiction; so f is a constant on A.

(b) Let $A = \mathbb{R}$ equipped with the Euclidean topology, and let $B = \{0, 1\}$, equipped with the indiscrete topology. Consider the function $f : A \to B$ defined by

$$f(x) = \begin{cases} 0 & : \text{ if } x = 0, \\ 1 & : \text{ otherwise.} \end{cases}$$

Let $D = \mathbb{R} \setminus \{0\}$. Clearly D is dense in A and f is constant on D. Moreover, the function f is continuous (since there are only two open sets in B, namely B and $\varnothing$, and their preimages under f are both open, being A and $\varnothing$ respectively). But f is not constant on A.

4. (5 marks)

(a) Given any open cover of a set of n elements, we choose the first set of the finite subcover to cover the first element, the second, to cover the second element, etc.

(b) Given any open cover, take any point and an open set U containing it. It contains the whole space with a finite number of points missing. By (a) they can be covered by a finite subcover.

(c) Any Euclidean open set is Sorgenfrey open by definition. Proving the contrapositive, we assume that K is not Euclidean compact. Then there is a Euclidean open cover for K with no open subcover. The same cover is Sorgenfrey open, so K is not Sorgenfrey compact.

(d) Take the open cover $[0, \frac{1}{2}) \cup [\frac{1}{2}, \frac{1}{3}) \cup [\frac{1}{3}, \frac{1}{4}) \cup \cdots \cup [1, 2)$. Every point of $[0, 1]$ is covered by only one member of the cover, so there is no finite subcover.

(e) Given any open cover, let U be the open set from it which covers $\{0\}$. Then it covers all the points of the set apart from a finite number of points. The claim then follows from (a).

(f) Let K be compact. For every $x \in K$, $\cup_n(-\infty, x - \frac{1}{n}) \cup [x, \infty)$ is a cover, so there is an ε such that $(x - \varepsilon, x) \cap K = \varnothing$. Choose a rational point in $(x - \varepsilon, x)$.

Bonus questions

1. (4 marks)

(a) This follows from homogeneity and the triangle inequality.

(b) Let $C_2 = n \max_i \|e_i\|$. By the projection inequalities, $|x_i| \leq \|x\|_0$, so from (a) $\|x\| \leq C_2 \|x\|_0$.

(c) Suppose $U \in \mathbb{R}$ is open and let $x \in \|\cdot\|^{-1}(U)$ (that is, $\|x\| = a \in U$). Then U contains the Euclidean ball (interval) $B_\varepsilon(a)$, for some $\varepsilon > 0$. Then for all $y \in \mathbb{R}^n$ with $\|y\|_0 < C_2^{-1}\varepsilon$ we have $|\|x + y\| - \|x\|| \leq \|y\| \leq C_2\|y\|_0 < \varepsilon$, so $\|\cdot\|^{-1}(U)$ contain the Euclidean ball $B_{C_2^{-1}\varepsilon}(x)$, and hence $\|\cdot\|^{-1}(U)$ is open.

(d) Because.

(e) By the Weierstrass Theorem, the continuous, positive function $\|\cdot\|$ attains its minimum $C_1 > 0$ on the Euclidean unit sphere in $\mathbb{R}^n$. Then $\|x\| \geq C_1$ for all x such that $\|x\|_0 = 1$ and the claim follows by homogeneity.

(f) By (b) and (e), any open ball in $\|\cdot\|$ contains a Euclidean open ball and vice versa, so that a set for $\|\cdot\|$ is open if and only if it is Euclidean open (the identity map is a homeomorphism). So the topology defined by any norm $\|\cdot\|$ coincides with the Euclidean topology.

2. (2 marks) For x on the equator, let $T(x)$ be the temperature at x and let $-x$ be the antipodal point. Apply the Intermediate Value Theorem to the function $f(x) = T(x) - T(-x)$.