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1. Return your assignment solutions to the MAT4TOP assignment box on level 3, PS2, by 2pm on September 16th.
2. Please start with the following statement of originality, which must be signed by you: `"This is my own work. I have not copied any of it from anyone else."`
3. The total of this assignment is 17 marks (not counting the Bonus Questions).

1. (Bases)

Prove Lemmas 2, 3 and 4 on page 11 of Lecture 10.

2. (Closure, interior, boundary)

Consider $\mathbb{R}$ with the half interval topology and let $B_1 = (0, 1]$, $B_2 = [0, 1)$. Determine the closure, the boundary and the interior of B_1 and B_2.

3. (Homogeneous spaces: a torus)

Denote $\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}$.

(a) Prove that $\mathbb{Z}^2$ is a group by addition.

(b) Define the *action* of the group $\mathbb{Z}^2$ on the topological space $\mathbb{R}^2$ as follows: for $g = (m, n) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2$, set $g((x, y)) = (x + m, y + n)$. Prove that this map is a homeomorphism, for every $g \in \mathbb{Z}^2$.

(c) Define the *orbit* of a point $(x, y) \in \mathbb{R}^2$ to be the set $\{g((x, y)) : g \in \mathbb{Z}^2\}$. The partition of $\mathbb{R}^2$ into the orbits defines the corresponding quotient space which is called a *torus* and is conventionally denoted $\mathbb{R}^2/\mathbb{Z}^2$.
Consider the map $f : \mathbb{R}^2 \to S^1 \times S^1$, $f((x, y)) = (e^{2\pi ix}, e^{2\pi iy})$ and use Lemma 1 of Lecture 9 to prove that there is a continuous bijection $h : \mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$. (*)

(d) Define a group G and its action on the open strip $\mathbb{R} \times (-1, 1) = \{(x, y) : |y| < 1\}$ by homeomorphisms such that $\mathbb{R}^2/G$ is the open Möbius band. Just the answer.

4. (Products and Hausdorffity)

(a) Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be Hausdorff spaces. Prove that $A \times B$ is Hausdorff.

(b) Let $(A, \mathcal{S})$ be a topological space and consider $A \times A$ equipped with the product topology. Let $\triangle$ denote the diagonal map: $\triangle : A \to A \times A$, $x \mapsto (x, x)$. Prove that A is Hausdorff if and only if the image of $\triangle$ is closed in $A \times A$.

Bonus questions

1. (4 marks) In Lecture 8 we proved that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq 1$. In this question we show that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

To prove this you can follow the plan below.

(*) Considering the images of basic rectangles it's easy to prove that h is also open, so $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$.

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(a) Explain why it suffices to show that there is no injective continuous map ϕ from the sphere S^2 to the plane $\mathbb{R}^2$. [(†)]

(b) Arguing by contradiction, suppose such a map exists. Explain why it is a homeomorphism on its image. Explain why $\phi(S^2)$ is compact.

(c) Show that the sphere is the union of two subsets homeomorphic to $\mathbb{R}^2$ and explain why any subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{R}^2$ is open.

(d) Deduce a contradiction from (b) and (c), hence proving the claim.

(e) In contrast with what we proved, construct a topological space $(A, \mathcal{S})$ such that $A^n \simeq A$, for all $n \in \mathbb{N}$.

2. **(3 marks)** Give an example of three topological spaces A_1, A_2 and B such that $A_1 \times B$ and $A_2 \times B$ are homeomorphic, but A_1 and A_2 are not. A careful proof is not necessary, but please give some convincing explanation.

[(†)] It's much more interesting: for any continuous map from the sphere S^2 to the plane $\mathbb{R}^2$, there is always a pair of antipodal points on S^2 whose images are the same: "there are two antipodal points on the surface of the Earth with the same temperature and the same air pressure".