A

MAT4TOP **Solutions to Assignment 3, 2015**

1. (6 marks)

Lemma 2 (Let $(A, \mathcal{S})$ have base $\mathcal{B}$ and let $B \subset A$. Then $x \in \overline{B}$ if and only if every element of $\mathcal{B}$ containing x intersects B). By definition, $(x \in \overline{B}) \Leftrightarrow (\forall U \in \mathcal{S},\ x \in U \Rightarrow U \cap B \neq \varnothing)$. In particular, this is true for $U \in \mathcal{B} \subset \mathcal{S}$, which establishes the "only if" part. To prove the "if" part, suppose that $V \in \mathcal{S}$ and $x \in V$. Then V is a union of elements from $\mathcal{B}$ and one of them, U, contains x. Then $U \cap B \neq \varnothing$ which implies $U \cap B \neq \varnothing$.

Lemma 3 (Let $(A, \mathcal{S})$ and $(B, \mathcal{T})$ be spaces and let $\mathcal{B}$ be a base for $\mathcal{T}$. A map $f : A \to B$ is continuous if and only if $f^{-1}(U) \in \mathcal{S}$ for every $U \in \mathcal{B}$). Again the "only if" follows from the fact that $\mathcal{B} \subset \mathcal{T}$. To prove the converse, take $V \in \mathcal{T}$. Then $V = \cup_{i \in I} U_i,\ U_i \in \mathcal{B}$. Then by Lemma3(1) of Lecture 1, $f^{-1}(V) = \cup_{i \in I} f^{-1}(U_i) \in \mathcal{S}$, as required.

Lemma 4 (Let $(A, \mathcal{S})$ be a topological space with base $\mathcal{B}$. Then B is dense in A if and only if $U \cap B \neq \varnothing$ for every non-empty $U \in \mathcal{B}$). And again, the "only if" follows from the fact that $\mathcal{B} \subset \mathcal{S}$. To prove the converse, take a non-empty $V \in \mathcal{S}$. Then $V = \cup_{i \in I} U_i,\ U_i \in \mathcal{B}$. As $I \neq \varnothing$, there is at least one U_i, which has something in common with B, and hance $V \cap B \neq \varnothing$, as required.

2. (3 marks)

The answers are: $\overline{(0,1]} = [0,1]$, $(0,1]^0 = (0,1)$ and $\partial(0,1] = \{0,1\}$. Here is the justification. First, $[0,1]$ is closed (since its complement is open) and so $\overline{(0,1]} \subset [0,1]$, by Theorem 1 of Lecture 7. On the other hand, for all $x \in [0,1]$ we have $[x, x+r) \cap (0,1] \neq \varnothing$, for all $r > 0$, and so if U is open and $x \in U$, then $U \cap (0,1] \neq \varnothing$. Hence $[0,1] \subset \overline{(0,1]}$, and so $\overline{(0,1]} = [0,1]$. Second, the set $(0,1)$ is open, so $(0,1) \subset (0,1]^0$. However, no open set containing 1 is contained in $(0,1]$, so $1 \notin (0,1]^0$. Hence $(0,1]^0 = (0,1)$. Finally, by Theorem 4 of Lecture 7, we have $\partial(0,1] = [0,1] \backslash (0,1) = \{0,1\}$.

Repeating the arguments, we obtain $\overline{[0,1)} = [0,1)^0 = [0,1)$ and $\partial[0,1) = \varnothing$.

3. (4 marks)

(a) Associativity follows from that in $\mathbb{R}$; the identity element is $(0,0)$; the reciprocal to (m,n) is $(-m,-n)$.

(b) The topology on $\mathbb{R}^2$ is define by the Euclidean metric, of which parallel translations are isometries.

(c) By Lemma 1 of Lecture 9, there is a well-defined, continuous and onto map $h : \mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$, whose value on the orbit of (x,y) equals $f((x,y))$. The map h is also one-to-one, as $f((x,y)) = f((x',y'))$ implies that (x,y) and (x',y') lie in the same orbit.

(d) $G = \mathbb{Z}$ acting like that: $n((x,y)) = (x+n, (-1)^n y)$, for $n \in \mathbb{Z}$.

4. (4 marks)

(a) Let (x,y) and (s,t) be distinct elements of $A \times B$. Then we must have $x \neq s$ or $y \neq t$. Consider the case where $x \neq s$. Since A is Hausdorff, there are disjoint open subsets U and V of A such that $x \in U$ and $s \in V$. By definition of the product topology $U \times B$ and $V \times B$ are disjoint, basic open subsets of $A \times B$ and $(x,y) \in U \times B$ and $(s,t) \in V \times B$.

(b) First suppose that the image $\triangle(A)$ is closed in $A \times A$. Let $a, b \in A$ with $a \neq b$. Then $(a,b) \notin \triangle(A)$. As $(A \times A) \setminus \triangle(A)$ is open, there is a basic open set $U \times V \subset A \times A$ such that $(a,b) \in U \times V$ (so that $a \in U$ and $b \in V$) and $(U \times V) \cap \triangle(A) = \varnothing$, which implies that $U \cap V = \varnothing$. Hence A is Hausdorff.

Conversely, suppose that A is Hausdorff. Let $(a,b) \in (A \times A) \setminus \triangle(A)$. Since A is Hausdorff, there exist disjoint open sets $U, V \subset A$ with $a \in U$, $b \in V$. Then $U \times V$ is open in $A \times A$ and $(U \times V) \cap \triangle(A) = \varnothing$. Hence $(A \times A) \setminus \triangle(A)$ is open and so $\triangle(A)$ is closed.

Bonus questions

1. (4 marks)

(a) Because if there is a homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^2$, then its restriction to $S^2 \subset \mathbb{R}^3$ is continuous and injective.

(b) $\phi : S^2 \to \phi(S^2)$ is a homeomorphism by Theorem 2 of Lecture 8 (plus Heine-Borel and the fact that any subset of $\mathbb{R}^2$ is Haudorff). $\phi(S^2)$ is compact by Theorem 1 of Lecture 6.

(c) The sphere minus a point is homeomorphic to $\mathbb{R}^2$ by the stereographic projection. So the sphere is the union of two subsets homeomorphic to $\mathbb{R}^2$. If ψ is a homeomorphism from $A \subset \mathbb{R}^2$ to $\mathbb{R}^2$, then for any $x \in A$, the set $\psi^{-1}(B_1(\psi(x)))$ is open and is contained in A. So A is open.

(d) From (c), $\phi(S^2)$ is open, while from (b) (and Heine-Borel) it is closed and bounded. So $\phi(S^2) \neq \varnothing, \mathbb{R}^2$ and is clopen. This contradicts the connectedness of $\mathbb{R}^2$ (Theorem 2 of Lecture 5; or, more argumented, Theorem 1 of Lecture 5 plus Theorem 3 of Lecture 11).

(e) Let $A = \mathbb{N}$ with discreet topology. Then A^n is countable, and with discrete topology, so $A^n \simeq A$, for all $n \in \mathbb{N}$.

2. (3 marks) Take A_1 to be the closed cylinder $S^1 \times [0,1]$, A_2 the closed Möbius band and B the closed interval $[0,1]$. Then both $A_1 \times B$ and $A_2 \times B$ are the solid tori (you may want to give some explanation), but A_1 and A_2 aren't homeomorphic.