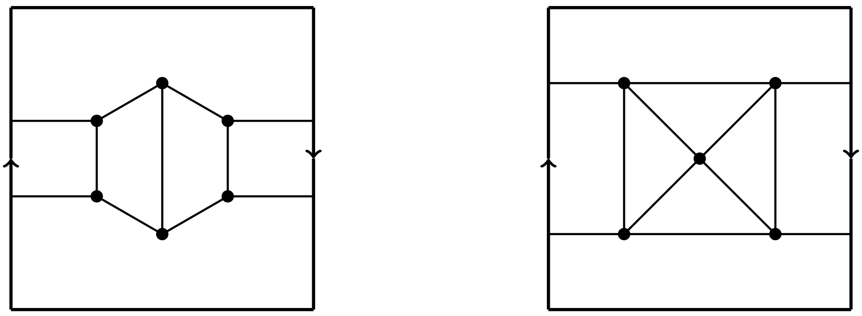
A

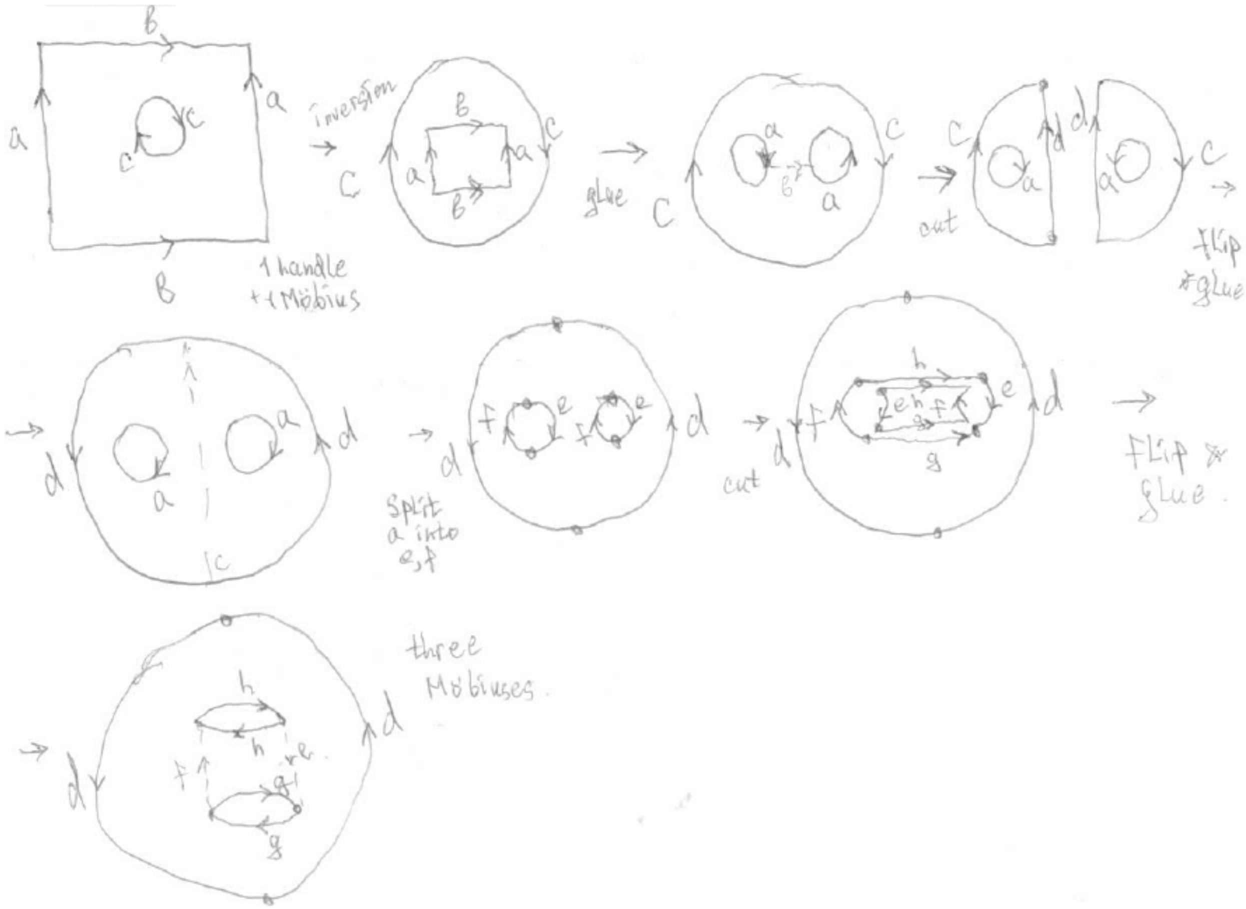
MAT4TOP **Solutions to Assignment 4, 2015**

1. (4 marks)

Drawings of $K_{3,3}$ (left) and of K_5 (right), with no edge crossings, on the Möbius band:



2. (4 marks)



3. (4 marks)

We have to prove that "the product operation respects homotopy classes, that is, if $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $f_0 \circ g_0 \simeq f_1 \circ g_1$."

Suppose $F(s,t)$ is a homotopy between f_0 and f_1, and $G(s,t)$ is a homotopy between g_0 and g_1. Define $H : [0,1] \times [0,1] \to A$ by

$$H(s,t) = \begin{cases} F(2s,t) & : \ s \in [0, \frac{1}{2}], \\ G(2s-1,t) & : \ s \in [\frac{1}{2}, 1]. \end{cases}$$

Such an H is continuous by Theorem 5 of Lecture 4, with $H(\cdot, 0) = f_0 \circ g_0$, $H(\cdot, 1) = f_1 \circ g_1$, and with $H(0,t) = f_t(0)$ and $H(1,t) = g_t(1)$ fixed. So H is a homotopy between $f_0 \circ g_0$ and $f_1 \circ g_1$.

4. (5 marks)

Let $x_0 \in X$, $y_0 \in Y$. For a loop $f : [0,1] \to X$ based at x_0 and a loop $g : [0,1] \to Y$ based at y_0, we define the loop $(f \times g) : [0,1] \to X \times Y$ based at (x_0, y_0) by $(f \times g)(s) = (f(s), g(s))$. Note that the map $\times$ so defined is surjective by Theorem 1 of Lecture 11. Moreover, for two loops f, f' in X based at x_0 and two loops g, g' in Y based at y_0, we have the following:

$$(f \times g) \simeq (f' \times g') \Leftrightarrow (f \simeq f' \text{ and } g \simeq g'). \tag{1}$$

Indeed, if F is a homotopy in X which joins f with f', and G is a homotopy in Y which joins g with g', then the map $H : [0,1] \times [0,1] \to X \times Y$ defined by $H(s,t) = (F(s,t), G(s,t))$ is continuous by Theorem 2 of Lecture 11 and joins the loops $f \times g$ and $f' \times g'$ in $X \times Y$ based at (x_0, y_0). Conversely, if a homotopy $H(s,t) = (F(s,t), G(s,t))$ joins the loops $f \times g$ and $f' \times g'$, then F joins f to f' and G joins g to g' by Theorem 1 of Lecture 11.

We define the product of corresponding elements from $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ by $[f] \otimes [g] = [f \times g] \in \pi_1(X \times Y, (x_0, y_0))$. We have to check few things.

- The map $\otimes : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ is well-defined. This follows from the "$\Leftarrow$" implication of (1).
- The map $\otimes : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ is surjective. This follows from the above.
- The map $\otimes$ is a homomorphism. We need to check that for $[f], [f'] \in \pi_1(X, x_0)$ and $g, g' \in \pi_1(Y, y_0)$, we have $([ff'] \otimes [gg']) = ([f] \otimes [g])([f'] \otimes [g'])$. This follows from the fact that both sides are represented by the loop in $X \times Y$ based at (x_0, y_0) given by

$$s \mapsto \begin{cases} (f(2s), g(2s)) & : \ s \in [0, \frac{1}{2}], \\ (f'(2s-1), g'(2s-1)) & : \ s \in [\frac{1}{2}, 1]. \end{cases}$$

- The map $\otimes : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ is injective. This follows from the "$\Rightarrow$" implication of (1).

It follows that $\otimes$ is a group isomorphism. Finally, by path connectedness of X, Y and $X \times Y$ (the latter follows from the Exercise after Theorem 3 of Lecture 11), the groups don't depend on the basepoints.

5. (3 marks)

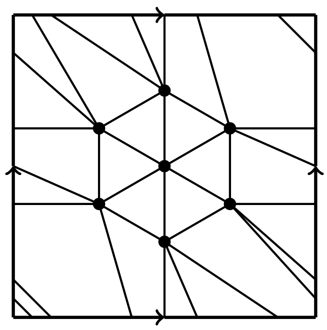
We represent the Möbius band M as the quotient space $([0,1] \times [-\frac{1}{2}, \frac{1}{2}])/\sim$, where $\sim$ identifies the points $(0,y)$ and $(1,-y)$ for $y \in [-\frac{1}{2}, \frac{1}{2}]$. Let the circle $S^1 = ([0,1] \times \{0\})/\sim$

be the middle line of M and let $\pi : M \to S^1$ be the natural projection (which descends from the coordinate projection $(x, y) \mapsto x$ through $\sim$). Let $p_0 = \{(0,0)\}/\sim \in S^1 \subset M$. We show that $\pi_1(M, p_0) = \pi_1(S^1, p_0) = \mathbb{Z}$.

Given a loop $f : [0,1] \to M$ based at p_0, let $f' = \pi \circ f$. Then f' is a loop at S^1 based at p_0. What is more, $f \simeq f'$: the homotopy is "squeezing" f to f' in the vertical direction. The map $f \mapsto f'$ is is clearly surjective. What is more, it descends to the map ϕ of the corresponding homotopy classes (defined by $\phi([f]) = [f'])$. Indeed, if we suppose that $f \simeq g$ in M, then $f' \simeq g'$ in M, which implies that $f' \simeq g'$ in S^1, because for a homotopy F in M joining f' and g', the homotopy $\pi \circ F$ joins f' and g' in S^1 (continuity of the composite). It follows that $\phi : \pi_1(M, p_0) \to \pi_1(S^1, p_0)$ is well-defined and surjective. It is also injective: if f and g are two loops in M such that $f' \simeq g'$ in S^1, then $f' \simeq g'$ in M (by the same homotopy), and so $f \simeq g$. So ϕ is a bijection. The fact that ϕ respects the product follows from the fact that the homotopy equivalence does (Lemma 3 of Lecture 15; see Question 3).

Bonus questions

1. (3 marks) K_7 is in fact toric (your drawing could be much nicer):



2. (3 marks) We have a continuous map (the natural projection) $\pi : S^2 \to \mathbb{R}P^2$ which identifies the antipodal points (see Lecture 9, including Example 10). Choose $x_0 \in \mathbb{R}P^2$ and denote $y_0 \in S^2$ to be one of the two points in $\pi^{-1}(x_0)$. Let $f : [0,1] \to \mathbb{R}P^2$ be a loop based at x_0. We lift it piece-by-piece to a path $\tilde{f}$ in S^2 starting at y_0, similar to how we lifted a path in a circle to that in $\mathbb{R}^2$ in the proof of Theorem 3 of Lecture 15. Let $y_1 = \tilde{f}(1)$. We have $\pi(y_1) = x_0$, so either $y_1 = y_0$, or $y_1 = -y_0$, the antipodal point. In the first case, $\tilde{f}$ is a loop on the sphere, and as such is homotopic to the trivial loop $c(s) = y_0$. If F is a homotopy which joins $\tilde{f}$ to c, then the homotopy $\pi \circ F$ joins f to the trivial loop on $\mathbb{R}P^2$. Hence such an f is null-homotopic in $\mathbb{R}P^2$. Consider the case when $\tilde{f}(1) = y_1$. We claim that $[f]$ is nontrivial. For if $[f] = 0$ we could have lifted the corresponding homotopy F to a homotopy $\tilde{F} = \tilde{F}(s,t)$ on S^2, with the point $\tilde{F}(1,t)$ changing continuously, which contradicts the fact that $\tilde{F}(1,0) = -y_0$, but $\tilde{F}(1,1) = y_0$ (alternatively, if such an F existed, then for t close to 1 the loop f_t would lie in a small neighbourhood U of x_0, for which $\pi^{-1}(U)$ is disconnected, with one component containing y_0 and another one $-y_0$).

Now if f' is any other loop based at x_0 whose lifted path $\tilde{f}'$ ends in $-y_0$, then $f \circ (f')^{-1}$ is a loop based at x_0 whose lift to S^2 ends in x_0, and so $[f][f']^{-1} = e$. It follows that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$.